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Symmetries of photorefractive four-wave mixing

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A symmetry analysis of degenerate four-wave-mixing equations in photorefractive crystals is carried out. Using underlying SU symmetries, a systematic derivation of conserved quantities is performed, and a method of integration of the equations is introduced. Five conserved quantities are found, suggesting that the initial four complex equations can be expressed in terms of three real quantities (Euler angles or other). However, due to the form of the equations, only one independent variable is found necessary to solve the problem.

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Four-wave-mixing (4WM) equations in photorefractive (PR) crystals are the most important equations in the field of optical phase conjugation (OPC). They have been solved up to now in a number of ways [1–4]. However, no systematic method has been offered up to now. Usually, conserved quantities are determined first, and then the number of equations is reduced. As a rule, this is accomplished in an *ad hoc* manner. Furthermore, the solution of the two most important geometrical arrangements (transmission and reflection) is performed most often using unrelated methods.

Utilizing underlying symmetries of wave equations, we offer a unified method, which systematically derives conserved quantities and reduces the number of independent equations to only one. This is an improvement over the known methods [1–4], which in general reduce the initial set of four complex equations to either four real (for intensities) or two complex (for amplitude ratios), and lead to rather involved solution procedures. The method can be applied to other wave-mixing processes as well.

Another symmetry procedure has been introduced recently by Barrett, Powell, and Hall [5] for the transmission 4WM. They were first to recognize SU(2) as the symmetry group of the problem. In a lucid application of the reciprocity theorem, Gu and Yeh have shown recently [6] that some conserved quantities for both transmission and reflection 4WM can be connected with the reciprocity principle of wave scattering.

The process of interest is the degenerate four-wave mixing in photorefractive crystals, represented in Fig. 1. In this process the crystal is illuminated by two counterpropagating laser beams A_1 and A_2 and by the signal A_4. Due to nonlinear interaction of these fields ("scattering of light by light"), a fourth wave A_3 is generated inside the crystal, counterpropagating to the probe A_4 and constituting its phase-conjugate (time-reversed) replica. The process proceeds through two main scattering channels, known as the transmission geometry (TG) and the reflection geometry (RG).

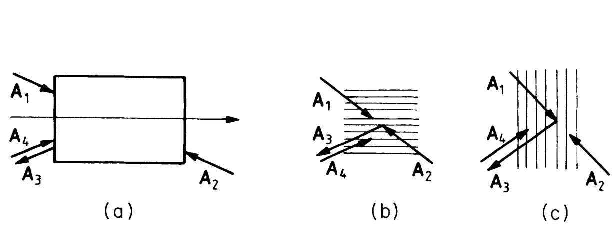

FIG. 1. (a) Four-wave-mixing geometry. (b) Transmission gratings. (c) Reflection gratings.

In TG it is presumed that the pump A_1 interferes with the signal A_4, building an interference pattern which (through the photorefractive effect) leaves an imprint in the refractive index of the crystal, in the form of a diffraction grating. The other pump is scattered off that grating into the phase-conjugate (PC) wave A_3 (and transmitted across the crystal during the process—thus, the name transmission geometry). In RG the grating is formed by the interference of the waves A_4 and A_2, and the pump A_1 is reflected off that grating into the PC reconstruction A_3.

These processes are described by the slowly-varying-envelope wave equations. For the TG, they are of the form [1]

$$IA_1' = \Gamma A_T A_4, \quad IA_4' = -\Gamma \bar{A}_T A_1 , \tag{1a}$$

$$IA_2' = \Gamma \bar{A}_T A_3, \quad IA_3' = -\Gamma A_T A_2 , \tag{1b}$$

where $I = \Sigma |A_i|^2$ is the total intensity, Γ is the coupling constant (a real number here), $A_T = A_1 \bar{A}_4 + \bar{A}_2 A_3$ is the transmission-grating amplitude, the prime denotes a spatial derivative along the propagation direction, and the bar denotes complex conjugation. Similarly, for the reflection grating,

$$IA_1' = -\Gamma A_R A_3, \quad IA_3' = -\Gamma \bar{A}_R A_1 , \tag{2a}$$

$$IA_2' = -\Gamma \bar{A}_R A_4, \quad IA_4' = -\Gamma A_R A_2 , \tag{2b}$$

where $A_R = A_1 \bar{A}_3 + \bar{A}_2 A_4$ is the reflection-grating amplitude. First we consider TG.

By defining the fundamental representation

$$|\psi_1\rangle = \begin{bmatrix} A_1 \\ A_4 \end{bmatrix}, \quad |\psi_2\rangle = \begin{bmatrix} A_3 \\ -A_2 \end{bmatrix} , \tag{3}$$

"equations of motion" (1) can be written using matrices:

$$|\psi_1\rangle' = im|\psi_1\rangle , \quad |\psi_2\rangle' = im|\psi_2\rangle \tag{4}$$

where

$$m=\begin{bmatrix} 0 & -i\mu \\ i\bar{\mu} & 0 \end{bmatrix} , \tag{5}$$

is a traceless and Hermitian matrix and $\mu=\Gamma A_T/I$. Such matrices belong to a representation of the SU(2) algebra [7]. Scalar products $\langle\psi|\psi\rangle$ define "integrals of motion." Thus

$$Q_1=\langle\psi_1|\psi_1\rangle=I_1+I_4 \quad \text{and} \quad Q_2=\langle\psi_2|\psi_2\rangle=I_3+I_2 , \tag{6}$$

and

$$C_2=\langle\psi_2|\psi_1\rangle=A_1\bar{A}_3-A_4\bar{A}_2 \tag{7}$$

are conserved quantities, as was known before [1]. Another conserved quantity is found if one recalls a simple result from quantum mechanics.

An evolution equation,

$$|\psi\rangle'=m_\psi|\psi\rangle , \tag{8}$$

has an integral of motion $\langle\psi|n|\phi\rangle$, if there exists a constant matrix n such that

$$nm_\phi+m_\psi^\dagger n=0 , \tag{9}$$

where the dagger denotes the adjoint matrix.

Having in mind the form of the matrix (5), the only possibility is

$$C_1=\langle\bar{\psi}_2|i\sigma_2|\psi_1\rangle=A_1A_2+A_3A_4 , \tag{10}$$

where σ_2 is one of the Pauli matrices. This quantity is also known from before [1].

The physical meaning of Q_1 and Q_2 is clear: They represent energy conservation. The meaning of C_1 and C_2 is not as obvious, and recently [6] they have been shown to be the consequence of the reciprocity theorem [8]. This theorem connects scattering matrices for the direct- and the reverse-scattering processes.

The same conserved quantities can also be found by considering irreducible or reducible representations, formed from the direct products of ψ_1 and ψ_2: $\psi_1\otimes\bar{\psi}_1$, $\psi_2\otimes\bar{\psi}_2$, $\psi_1\otimes\psi_2$, $\psi_1\otimes\bar{\psi}_2$, etc. For example, if one considers reducible representations $\psi\otimes\bar{\psi}_1$ and $\psi_2\otimes\bar{\psi}_2$, a triplet and a singlet irreducible representation are found, in which the singlets are conserved:

$$\psi_1\otimes\bar{\psi}_1=\begin{bmatrix} A_1\bar{A}_4 \\ A_4\bar{A}_4-A_1\bar{A}_1 \\ A_4\bar{A}_1 \end{bmatrix}+Q_1 \tag{11a}$$

and

$$\psi_2\otimes\bar{\psi}_2=\begin{bmatrix} A_3\bar{A}_2 \\ A_2\bar{A}_2-A_3\bar{A}_3 \\ A_2\bar{A}_3 \end{bmatrix}+Q_2 . \tag{11b}$$

When $\psi_1\otimes\psi_2$ and $\psi_1\otimes\bar{\psi}_2$ are reduced, the remining conserved singlets emerge

$$\psi_1\otimes\psi_2=\begin{bmatrix} A_1A_3 \\ A_4A_3-A_1A_2 \\ A_4A_2 \end{bmatrix}+C_1 \tag{12a}$$

and

$$\psi_1\otimes\bar{\psi}_2=\begin{bmatrix} A_1\bar{A}_2 \\ A_4\bar{A}_2+A_1\bar{A}_3 \\ A_4\bar{A}_3 \end{bmatrix}+C_2 . \tag{12b}$$

Thus, there exist four conserved quantities: two real ones (Q_1 and Q_2) and two complex (C_1 and C_2) ones. However, not all of them are independent. It is easy to check that

$$|C_1|^2+|C_2|^2=Q_1Q_2 , \tag{13}$$

and so there are five real constants in all. Correspondingly, the initial set of equations could be expressed in terms of three variables—for example, Euler angles or quaternions. However, we will demonstrate that, in fact, only one independent variable is necessary to solve the problem.

Thus far, only linear aspects of the symmetry of equations have been utilized. When the nonlinear, field-dependent form of the matrix elements is explored, a further reduction in the number of variables is achieved. This reduction cannot be simply represented by a group of lower symmetry than SU(2). Instead, it is simpler to think in terms of constraints or connections between the variables, which bring the number of independent variables down to one.

To this end, let us find an evolution equation for A_T:

$$IA_T'=-\Gamma A_T\delta , \tag{14}$$

where $\delta=I_1+I_2-I_3-I_4$. The evolution equation for δ is

$$I\delta'=4\Gamma|A_T|^2 , \tag{15}$$

which yields another conserved quantity: $Q_3=\delta^2+4|A_T|^2$. This quantity is related to the ones already found.

The existence of conserved quantities Q_1 and Q_2 suggests a convenient choice of variables:

$$A_1=\sqrt{Q_1}\cos(\alpha_1)\exp(i\beta_1) , \tag{16a}$$

$$A_4=\sqrt{Q_1}\sin(\alpha_1)\exp(i\gamma_1) , \tag{16b}$$

$$A_2=\sqrt{Q_2}\cos(\alpha_2)\exp(i\beta_2) , \tag{16c}$$

$$A_3=\sqrt{Q_2}\sin(\alpha_2)\exp(i\gamma_2) , \tag{16d}$$

so that the new variables are six angles: α_1, α_2, β_1, β_2, γ_1, and γ_2. The new "phase space" can be visualized as a direct product of two 3-spheres, (α_1, β_1, γ_1) and (α_2, β_2, γ_2). These variables can also be written in terms of two sets of Euler angles. However, only one set is linearly independent. This follows from the fact that if $|A_1,A_4\rangle$ is a solution to Eqs. (1), then $|-\bar{A}_4,\bar{A}_1\rangle$ is also (an independent) solution. Any other solution (such as $|A_{3,}-A_2\rangle$) could then be written as a superposition, and only three Euler angles (at best) are independent. The integral Q_3 can be expressed in terms of the variables given by Eqs. (16), and the result is

$$Q_3=Q_1^2+Q_2^2+2Q_1Q_2[\cos(2\alpha_1)\cos(2\alpha_2) + \sin(2\alpha_1)\sin(2\alpha_2)\cos(\phi)] , \tag{17}$$

where $\phi=\beta_1+\beta_2-\gamma_1-\gamma_2$ is a so-called relative phase. This quantity is very important in OPC. When $\phi=0$ (or π, depending on the experimental setup), we have exact phase conjugation. In this case [cf. Eqs. (25)] the phases are constant.

From spherical trigonometry it is known that the expression in brackets in Eq. (17) can be understood as a cosine of some angle ρ, so that $2\alpha_1$, $2\alpha_2$, and ρ are the sides of a spherical triangle and ϕ is its central angle [9]. Therefore,

$$\cos(\rho)=\frac{Q_3-Q_1^2-Q_2^2}{2Q_1Q_2}=\text{const} , \tag{18}$$

and ϕ only depends on α_1 and α_2:

$$\cos(\phi)=\frac{\cos(\rho)-\cos(2\alpha_1)\cos(2\alpha_2)}{\sin(2\alpha_1)\sin(2\alpha_2)} . \tag{19}$$

Further, it can be shown that

$$|C_1|^2=Q_1Q_2\cos^2(\rho/2)$$

and (20)

$$|C_2|^2=Q_1Q_2\sin^2(\rho/2) ,$$

so that the complete list of five independent integrals reads as follows: Q_1, Q_2, Q_3 (or ρ), $\arg(C_1)$, and $\arg(C_2)$.

The equations for α_1 and α_2, extracted from Eqs. (1), form a closed system of equations:

$$2I\alpha_1'=-\Gamma[Q_1\sin(2\alpha_1)+Q_2\sin(2\alpha_2)\cos(\phi)] , \tag{21a}$$

$$2I\alpha_2'=-\Gamma[Q_1\sin(2\alpha_1)\cos(\phi)+Q_2\sin(2\alpha_2)] , \tag{21b}$$

which are solved by introducing two auxiliary variables,

$$x_1=\cos(2\alpha_1) \text{ and } x_2=\cos(2\alpha_2) . \tag{22}$$

Equations (21) then can be rewritten as

$$Ix_1'=\Gamma\{[Q_1+Q_2\cos(\rho)]-x_1\delta\} , \tag{23a}$$

$$Ix_2'=\Gamma\{[Q_1\cos(\rho)+Q_2]-x_2\delta\} , \tag{23b}$$

where $\delta=Q_1x_1+Q_2x_2$ is found from Eq. (15):

$$I\delta'=\Gamma(Q_3-\delta^2) . \tag{24}$$

With δ known, Eqs. (23) are reduced to quadratures. Then δ is the only variable needed to solve the problem. Once α_1 and α_2 are known, the remaining four angles are found easily:

$$2I\beta_1'=-\Gamma Q_2\sin(\phi)\sin(2\alpha_2)\tan(\alpha_1) , \tag{25a}$$

$$2I\beta_2'=-\Gamma Q_1\sin(\phi)\sin(2\alpha_1)\tan(\alpha_2) , \tag{25b}$$

$$2I\gamma_1'=-\Gamma Q_2\sin(\phi)\sin(2\alpha_2)\cot(\alpha_1) , \tag{25c}$$

$$2I\gamma_2'=-\Gamma Q_1\sin(\phi)\sin(2\alpha_1)\cot(\alpha_2) . \tag{25d}$$

The problem, therefore, can be reduced to one equation and one variable. The other variables can be solved in quadratures. To complete the solution, it remains to fit boundary conditions.

In the remainder we point out how to solve RG using the same method. First, note that the "integrals of motion" in RG involving intensities now are the differences:

$$R_1=I_1-I_3 , \quad R_2=I_2-I_4 . \tag{26}$$

The total intensity is not conserved anymore. Let us find the equation for I:

$$II'=-4\Gamma|A_R|^2 . \tag{27}$$

Since

$$(|A_R|^2)'=-2\Gamma|A_R|^2 , \tag{28}$$

a new conserved quantity is obtained:

$$R_3=I^2-4|A_R|^2 . \tag{29}$$

Utilizing these integrals, a set of new variables is introduced:

$$A_1=\sqrt{R_1}\cosh(\alpha_1)\exp(i\beta_1) , \tag{30a}$$

$$A_3=\sqrt{R_1}\sinh(\alpha_1)\exp(i\gamma_1) , \tag{30b}$$

$$A_2=\sqrt{R_2}\cosh(\alpha_2)\exp(i\beta_2) , \tag{30c}$$

$$A_4=\sqrt{R_2}\sinh(\alpha_2)\exp(i\gamma_2) . \tag{30d}$$

The phase space is now a direct product of two hyperboloids. For simplicity, it is assumed that $R_1,R_2>0$. Using these variables, as in the transmission case, R_3 can be written as

$$R_3=R_1^2+R_2^2+2R_1R_2\cosh(\tau) , \tag{31}$$

where

$$\cosh(\tau)=\cosh(2\alpha_1)\cosh(2\alpha_2) - \sinh(2\alpha_1)\sinh(2\alpha_2)\cos(\phi) . \tag{32}$$

This expression is the content of the cosine theorem on a hyperbolic surface [9]. Thus τ is also conserved:

$$\cosh(\tau)=\frac{R_3-R_1^2-R_2^2}{2R_1R_2}=\text{const} , \tag{33}$$

and again, the relative angle ϕ can be expressed in terms of two real variables α_1 and α_2:

$$\cos(\phi)=\frac{\cosh(2\alpha_1)\cosh(2\alpha_2)-\cosh(\tau)}{\sinh(2\alpha_1)\sinh(2\alpha_2)} . \tag{34}$$

Equations for $x_1=\cosh(2\alpha_1)$ and $x_2=\cosh(2\alpha_2)$ are given by

$$Ix_1'=\Gamma[R_1+R_2\cosh(\tau)-Ix_1] \ , \tag{35a}$$

$$Ix_2'=\Gamma[R_2+R_1\cosh(\tau)-Ix_2] \ , \tag{35b}$$

where $I=R_1x_1+R_2x_2$. These equations can be solved once the equation for I is integrated [10]:

$$II'=\Gamma(R_3-I^2) \ . \tag{36}$$

Here the total intensity plays the role of the δ variable. The phases $\beta_{1/2}$ and $\gamma_{1/2}$ satisfy

$$2I\beta_1'=\Gamma R_2\sin(\phi)\sinh(2\alpha_2)\tanh(\alpha_1) \ , \tag{37a}$$

$$2I\beta_2'=\Gamma R_1\sin(\phi)\sinh(2\alpha_1)\tanh(\alpha_2) \ , \tag{37b}$$

$$2I\gamma_1'=\Gamma R_2\sin(\phi)\sinh(2\alpha_2)\coth(\alpha_1) \ , \tag{37c}$$

$$2I\gamma_2'=\Gamma R_1\sin(\phi)\sinh(2\alpha_1)\coth(\alpha_2) \ , \tag{37d}$$

and all of them can be found once α_1 and α_2 are known. Thus, in both TG and RG, only one real quantity is needed in order to solve the 4WM problem.

The corresponding group symmetry for the reflection geometry is deduced by considering a representation:

$$|\psi_1\rangle=\begin{bmatrix}A_1\\A_3\end{bmatrix} \ , \quad |\psi_2\rangle=\begin{bmatrix}A_4\\A_2\end{bmatrix} \ , \tag{38}$$

with the "dynamical" matrix:

$$m=\begin{bmatrix}0 & \mu\\ \bar{\mu} & 0\end{bmatrix} \ , \tag{39}$$

where $\mu=\Gamma A_R/I$. Such matrices are Hermitian and traceless, and form here an SU(1,1) group representation. Conserved quantities are found if one notes that the n matrix is now σ_3, so that the integrals are

$$R_1=\langle\psi_1|\sigma_3|\psi_1\rangle=I_1-I_3 \ ,$$
$$-R_2=\langle\psi_2|\sigma_3|\psi_2\rangle=I_4-I_2 \ . \tag{40}$$

Similary to the previous case, the products $\langle\psi_2|\sigma_3|\psi_1\rangle$ and $\langle\bar{\psi}_2|i\sigma_2|\psi_1\rangle$, or induced representations $\psi_1\otimes\psi_2$ and $\psi_1\otimes\bar{\psi}_2$ offer further two conserved singlets:

$$B_1=A_3A_4-A_1A_2 \quad \text{and} \quad B_2=A_3\bar{A}_2-A_1\bar{A}_4 \ . \tag{41}$$

In terms of the already defined variables and integrals, we have

$$|B_1|^2=R_1R_2\cosh^2(\tau/2)$$

and (42)

$$|B_2|^2=R_1R_2\sinh^2(\tau/2) \ ,$$

so that $|B_1|^2-|B_2|^2=R_1R_2$. As in the transmission case, there are five independent integrals of motion. Again, the amplitudes of B_1 and B_2 do not carry independent information; only their phases do.

In summary, we have investigated symmetries connected with the degenerate 4WM in PR crystals. It is noted that the solutions to wave equations can be written as representations of the SU(2) group in the case of the transmission geometry, and of the SU(1,1) group in the case of the reflection geometry. This fact is utilized to systematically derive conserved quantities in both cases and to present a method for integration of these equations. By using the form of "dynamical" equations and by deriving independent equations for the difference δ and the sum I of intensities, only one independent variable is found necessary to solve the whole problem.

[1] M. Cronin-Golomb, J. O. White, B. Fisher, and A. Yariv, Opt. Lett. **7**, 313 (1982); IEEE J. Quantum Electron. **QE-20**, 12 (1984).
[2] M. R. Belić, Phys. Rev. A **31**, 3169 (1985); M. R. Belić, and M. Lax, Opt. Commun. **56**, 197 (1985).
[3] A. Bledowski, W. Krolikowski, and A. Kujawski, IEEE J. Quantum Electron. **QE-22**, 1547 (1986); A. Bledowski and W. Krolikowski, Opt. Lett. **13**, 146 (1988).
[4] A. A. Zozulya and V. T. Tikhonchuk, Kvant. Elektron. (Moscow) **15**, 1570 (1988) [Sov. J. Quantum Electron. **18**, 981 (1988)].
[5] G. R. Barrett, A. K. Powell, and T. J. Hall, in *Proceedings of the Institute of Physics Applied Optics and Optoelectronics Conference, Nottingham, England, 1990* (IOP Publishing Ltd., Bristol, England 1990); D. A. Fish, A. K. Powell, and T. J. Hall (private communcation).
[6] C. Gu and P. Yeh, Opt. Lett. **16**, 455 (1991).
[7] J. P. Elliott and P. G. Dawber, *Symmetry in Physics* (MacMillan, London 1979).
[8] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1977); M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1980).
[9] M. Berger, *Geometrie* (Nathan, Paris, 1977).
[10] P. Stojkov and M. Belić (unpublished).