RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 37, NUMBER 5 MARCH 1, 1988

Phase conjugation via multiple gratings in photorefractive crystals

M. R. Belić
Max-Planck-Institut für Quantenoptik, D-8046 Garching, Federal Republic of Germany
and Institute of Physics, P. O. Box 57, 11001 Belgrade, Yugoslavia
(Received 13 July 1987)

We consider phase conjugation via degenerate four-wave mixing in photorefractive crystals when more than one grating mechanism is operative. A method for exact solution of the steady-state wave equations which include pump depletion is presented. It is discovered that equal-strength multigrating operation does not allow signal amplification.

Phase conjugation is now a mature subject of enough importance to find its way to textbooks,[1] special monographs,[2] review papers,[3] and *Physics Today*.[4] Its importance is certain to grow even further in view of its potential or realized applicability in various branches of nonlinear optics.[1-4] The basic principles of phase conjugation are well understood, and the differences between theoretical predictions and experimental results are ever decreasing. On the theoretical side, after an initial flurry of papers on the fundamentals and simple linearized theories,[2-5] people have settled now on evaluation of the more subtle effects and a more complete theory, such as the one which includes pump depletion,[6] absorption,[7] and different wave-mixing mechanisms.[8]

The problem with a more complete treatment is that it makes the theory strongly nonlinear, and consequently difficult to handle analytically. Completely analytical treatments do not seem to be feasible, and use of some numerics is unavoidable. In our method (denoted exact rather than analytic) an effort is made to reduce numerics to a minimum, and to relate different quantities analytically. Thus we present an exact analysis of phase conjugation via steady-state degenerate four-wave mixing (4WM) in photorefractive crystals when more than one volume grating is operative.

Our goal is rather limited. We will not be concerned with the physics of the conjugation process in any way, but with the solution of a system of coupled wave equations,[3] widely accepted to correctly represent this process in photorefractive dynamic media. At the end we will compare our results for multiple gratings with the known results when one grating is predominant.

The starting point is the following set of equations in the slowly varying amplitude approximation for the pump beams A_1 and A_2, the signal A_4, and its conjugate A_3, all plane waves.[2,3]

$$IA_1' = \gamma_T^* A_T A_4 - \gamma_R A_R A_3 - \gamma_P |A_2|^2 A_1 , \quad \text{(1a)}$$

$$IA_2^{*\prime} = \gamma_T^* A_T A_3^* - \gamma_R A_R A_4^* - \gamma_P |A_1|^2 A_2^* , \quad \text{(1b)}$$

$$IA_3' = -\gamma_T^* A_T A_2 - \gamma_R^* A_R^* A_1 - \gamma_S^* |A_4|^2 A_3 , \quad \text{(1c)}$$

$$IA_4^{*\prime} = -\gamma_T^* A_T A_1 - \gamma_R^* A_R^* A_2 - \gamma_S^* |A_3|^2 A_4 , \quad \text{(1d)}$$

where the prime denotes differentiation in the propagation z direction, and the asterisk denotes complex conjugation. I represents the total intensity, $I = \sum |A_j|^2$, while γ's represent different complex coupling constants, in principle, of the form $\gamma = in \exp(i\theta)$, where n's are material parameters (real numbers), and θ's give the spatial phase shift between the refractive-index gratings and the light interference pattern. Subscripts T,R,P,S stand for different grating mechanisms, i.e., different ways in which light beams can combine to build intensity interference patterns at allowed **k** vectors.

The assumed interaction geometry is the standard 4WM arrangement, in which two counterpropagating laser pumps illuminate the photorefractive crystal situated between the planes $z=0$ and $z=d$. From the side of the pump 1, and tilted for a small angle comes the signal A_4, and out of the medium, in the same direction, goes the phase conjugate A_3. The transmission grating is written by the interference fringes of the signal 4 with the pump 1, and by the phase conjugate 3 and the pump 2, which are also **k** matched. The corresponding wave-mixing term in the wave equations is $A_T = A_1 A_4^* + A_2^* A_3$. Analogously for the reflection grating, which is written by the beams 4 and 2, and 3 and 1, the wave-mixing term is $A_R = A_1 A_3^* + A_2^* A_4$. The P and S terms stand for the counterpropagating two-wave contributions, the first one $A_1 A_2^*$ coming from the pumps, and the second one $A_3 A_4^*$ coming from the signal and its conjugate.

When written in the form of intensities and the relative phase, Eqs. (1) become

$$\frac{II_1'}{2} = I_1 \sin\theta (I_2 n_P + I_3 n_R - I_4 n_T) + \sqrt{I_1 I_2 I_3 I_4}[n_R \sin(\phi+\theta) + n_T \sin(\phi-\theta)] , \quad \text{(2a)}$$

$$\frac{II_2'}{2} = I_2 \sin\theta (I_1 n_P - I_3 n_T + I_4 n_R) - \sqrt{I_1 I_2 I_3 I_4}[n_R \sin(\phi-\theta) + n_T \sin(\phi+\theta)] , \quad \text{(2b)}$$

$$\frac{II_3'}{2} = I_3 \sin\theta (I_1 n_R + I_2 n_T + I_4 n_S) + \sqrt{I_1 I_2 I_3 I_4}(n_R + n_T) \sin(\phi+\theta) , \quad \text{(2c)}$$

$$\frac{II_4'}{2}=I_4\sin\theta(I_1n_T+I_2n_R+I_3n_S)-\sqrt{I_1I_2I_3I_4}(n_R+n_T)\sin(\phi-\theta)\ , \tag{2d}$$

$$I\phi'=-\cos\theta[I_1(n_P-n_R+n_T)+I_2(n_R-n_T-n_P)+I_3(n_T-n_R+n_S)+I_4(n_R-n_T-n_S)]$$
$$-\sqrt{I_1I_2I_3I_4}\left[\cos(\phi+\theta)\left(\frac{n_T}{I_2}-\frac{n_R}{I_1}-\frac{n_T+n_R}{I_3}-\frac{n_T+n_R}{I_4}\right)+\cos(\phi-\theta)\left(\frac{n_R}{I_2}-\frac{n_T}{I_1}\right)\right]\ , \tag{2e}$$

where $A_j=\sqrt{I_j}\exp(i\phi_j)$ and $\phi=\phi_4+\phi_3-\phi_2-\phi_1$. In Eqs. (2) it is assumed that all gratings are shifted by the same amount θ with respect to the light interference pattern. In a moment we will restrict our attention to the most interesting case of photorefractive crystals, in which $\theta=\pi/2$. Further, we will only consider the case of exact phase conjugation $\phi=0$ (or $\phi=\pi$, depending on the experimental setup). In this case Eq. (2e) is trivially satisfied, and drops out of the picture. We are left with the four equations for the energy transfer,

$$II_1'=2I_1(I_2n_P+I_3n_R-I_4n_T)+2\sqrt{I_1I_2I_3I_4}(n_R-n_T)\ , \tag{3a}$$

$$II_2'=2I_2(I_1n_P-I_3n_T+I_4n_R)+2\sqrt{I_1I_2I_3I_4}(n_R-n_T)\ , \tag{3b}$$

$$II_3'=2I_3(I_1n_R+I_2n_T+I_4n_S)+2\sqrt{I_1I_2I_3I_4}(n_R+n_T)\ , \tag{3c}$$

$$II_4'=2I_4(I_1n_T+I_2n_R+I_3n_S)+2\sqrt{I_1I_2I_3I_4}(n_R+n_T)\ , \tag{3d}$$

with split boundary conditions: I_1 and I_4 are given on the $z=0$ face, and I_2 and $I_3=0$ on the $z=d$ face of the crystal. These equations reduce to the familiar case of the transmission geometry[3,5] when $n_R=0$, and to the reflection geometry[5,7] when $n_T=0$, and when, as usual, the two-wave terms are neglected. Our aim here is to investigate exactly the opposite case when the transmitive and the reflective gratings contribute about equally, and when the two-wave terms cannot be neglected. Consequently we will set $n_R=n_T=n\neq0$ and $n_S=n_P=m\neq0$; this simplified problem is still amenable to analytical treatment. The general case $n_R\neq n_T\neq0$, $n_S\neq n_P\neq0$ will be treated numerically in a subsequent publication.

In the next step a set of new variables is introduced:[7] $u_1=I_2+I_1$, $v_1=I_2-I_1$, $u_2=I_4+I_3$, $v_2=I_4-I_3$. When Eqs. (3) are written in terms of these new variables, it is seen that v_1 and v_2 are simply related: $v_1=v_2+\Delta$, where $\Delta=v_{1d}-v_{2d}$ is a constant evaluated at $z=d$. The remaining three equations have the form

$$Iu_1'=mf_1^2+2nv_1v_2\ , \tag{4a}$$

$$Iu_2'=mf_2^2+2n(u_1u_2+f_1f_2)\ , \tag{4b}$$

$$Iv_2'=2nu_1v_2\ , \tag{4c}$$

where $f_1^2=u_1^2-v_1^2=4I_1I_2$ and $f_2^2=u_2^2-v_2^2=4I_3I_4$ are some auxiliary functions. These functions obey differential equations of their own:

$$If_1'=mu_1f_1\ , \tag{5a}$$

$$If_2'=mu_2f_2+2n(u_1f_2+u_2f_1)\ , \tag{5b}$$

which, of course are not independent of Eqs. (4), but can be used to further simplify the problem. In fact, f_1 is also a simple function of v_2, $f_1/f_{1d}=(v_2/v_{2d})^{m/2n}$, and therefore, so is $u_1=[(v_2+\Delta)^2+f_{1d}^2(v_2/v_{2d})^{m/n}]^{1/2}$. Half of the variables are then represented in terms of v_2. There remain two equations to be solved: for u_2 and v_2, or for u_2 and f_2. These variables can also be represented in terms of v_2, and the most convenient choice is $u_2=v_2\cosh w$ and $f_2=v_2\sinh w$. The equations to be solved become

$$iv'=2nuv\ , \tag{6a}$$

$$iw'=2nf+mv\sinh w\ , \tag{6b}$$

where now all variables are scaled with respect to $v_{2d}=I_{4d}$, i.e., $v=v_2/v_{2d}$, $u=u_1/v_{2d}$, and $f=f_1/v_{2d}$. The total intensity is thus $i=u+v\cosh w$. Unfortunately, these two equations cannot be separated. As long as $m\neq0$ this set cannot be solved analytically. Numerical solution has to be performed, and the results are given and analyzed below. When $m=0$ the integration is easy, and proceeds as follows.

First, note that Eq. (6a) can be integrated formally for arbitrary value of m:

$$\ln v+\int_1^v\frac{dx}{[(x+\delta)^2+a^2x^b]^{1/2}}\cosh w(x)=2n(z-d)\ , \tag{7a}$$

where $\delta=\Delta/v_{2d}$, $a=f_{1d}/v_{2d}$, and $b=m/n$. Function $w(v)$ is to be evaluated from the other equation, or from a combination of the both. In fact, for $m=0$ the two equations can be divided, yielding

$$w(v)=\frac{a}{c}\ln\frac{v+\delta+(a-c)(u+a)}{v+\delta+(a+c)(u+a)} \tag{7b}$$

up to an integration constant. Here $c=(a^2+\delta^2)^{1/2}$. In general the quadrature in Eq. (7a) cannot be expressed in terms of simple functions. However, its numerical evaluation is quite simple, and its tabulation or graphical representation is easy.

With hindsight (cf. Fig. 1), we also note that w is nearly a linear function in z over a wide range of experimentally accessible parameters:

$$w\simeq\frac{2na}{[(1+\delta)^2+a^2]^{1/2}+1}(z-d)\ . \tag{8}$$

The cumbersome quadrature is then avoided, as Eq. (7b) immediately defines $v(z)$. This completes the solution procedure.

The choice of variables in Eqs. (6) is made so as to convert the original split-boundary value problem into an easier initial value problem. The initial values are given on the $z=d$ face, $v_d=1$, $w_d=0$, and the integration is performed backwards to $z=0$. The boundary value nature of the problem, however, is still retained in the explicit dependence of the parameters a and δ on the missing boundary values:

$$\delta=\frac{C_2-I_{1d}}{I_{4d}}-1,\quad a=\frac{2\sqrt{C_2I_1d}}{I_{4d}}\ , \tag{9}$$

where $C_2=I_{2d}$ is given, and I_{1d} and I_{4d} are to be evaluated. The evaluation can be performed in more than one way, and we opted here for a simple shooting procedure that is easy to implement numerically. In the method the missing boundary values are estimated at $z=d$, and the equations are integrated backwards to $z=0$, where the solution should equal the prescribed boundary values. The condition for equality gives a set of nonlinear equations for the estimated values, which is solved by Newton's method. A new set of improved estimates is thus obtained, and the procedure is iterated.

In Figs. 1–5 we present some of our results, illustrating the solution method and depicting some of the interesting features which characterize the multigrating 4WM operation in photorefractive crystals. In Fig. 1 a typical form of the functions v and w for a set of n,m values is shown, and in Fig. 2 the corresponding intensities are plotted. Each such graph defines a value of the intensity reflectivity on the $z=0$ face of the crystal, and many such values are collected and plotted in Figs. 3 and 4 as functions of the pump intensity ratio $r=I_{2d}/I_{10}$.

One striking feature is immediately apparent in Figs. 3 and 4—saturation of the reflectivity at $\rho=1.0$ for strong n coupling. This feature is characteristic of the equal strength multigrating operation, and is easy to understand. For a high value of n (or rather nd), the depletion of the probe is efficient, and its value at $z=d$ is small. Then the equations and boundary conditions for I_3 and I_4 are equivalent, making them practically indistinguishable and forcing the reflectivity to unity. In turn I_1 and I_2 be-

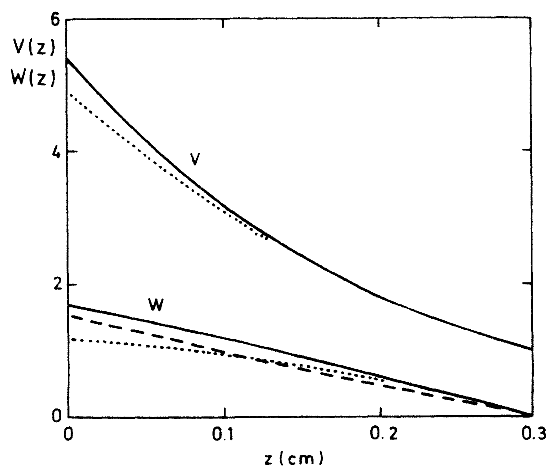

FIG. 1. Functions $v(z)$ and $w(z)$ for $n=-\sqrt{10}\simeq-3.16$, and for a set of values for m. Solid lines are for $m=0$, dashed lines for $m=-10^{2/3}\simeq-4.64$, and dotted lines for $m=4.64$. All numbers are given in units of cm^{-1}. The dashed line for v lies very close to the solid line for v and therefore is not plotted.

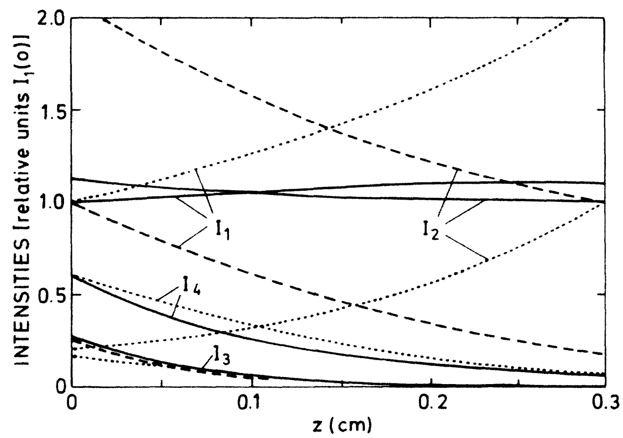

FIG. 2. Intensities of the four beams corresponding to the functions v and w from Fig. 1, and for the same values of n and m. It is seen that the probe I_4 and its phase conjugate I_3 are little sensitive to the variations in m, in contrast to the pumps.

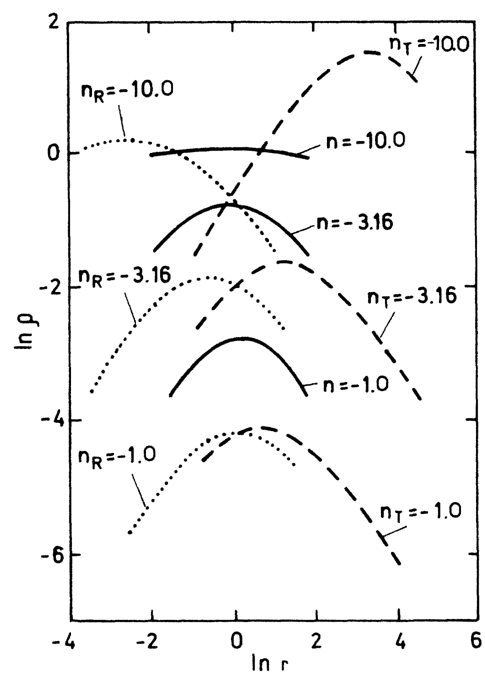

FIG. 3. Intensity reflectivity $\rho=I_{30}/I_{40}$ as a function of the pump ratio $r=I_{2d}/I_{10}$ for $m=0$. The solid curves represent the multigrating operation, the dashed represent the pure transmission grating, and the dotted the pure reflection grating. All the values for n, n_T, and n_R are given in cm^{-1}.

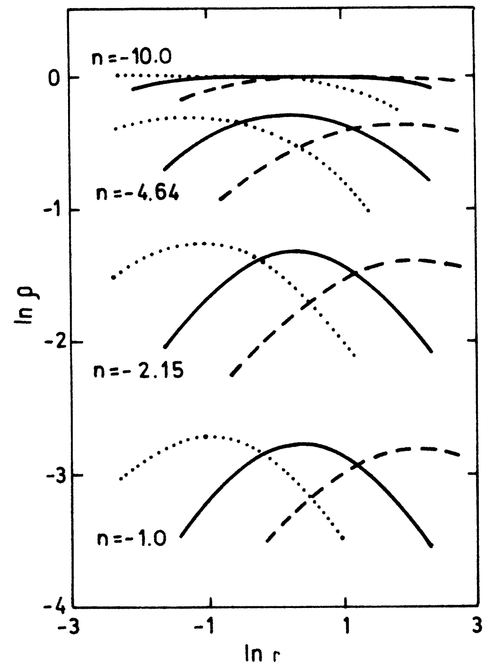

FIG. 4. Display of the effect of m on the reflectivity, for different values of n. The solid lines are for $m=0$, the dashed for $m=-4.64$, and the dotted for $m=4.64$. The saturation of the reflectivity at $\rho=1$ is clearly visible. In both this figure and Fig. 3 the probe intensity is kept fixed at $I_{40}=0.6I_{10}$.

come strongly m dependent (and n independent), being simply constant if $m=0$. In general, I_3 and I_4 are less sensitive to the variations in m, and I_1 and I_2 are less sensitive to the variations in n. Consequently, functions v and w are only a little sensitive to the variations in m, and a good first guess are the expressions for $m=0$.

In Fig. 5 the effect of the probe ratio $q=I_{40}/(I_{10}+I_{2d})$ on the reflectivity is depicted. It is seen that this effect is not very pronounced, consisting in a slight upshift in the middle of the reflectivity curve, and a slight downshift in the wings.

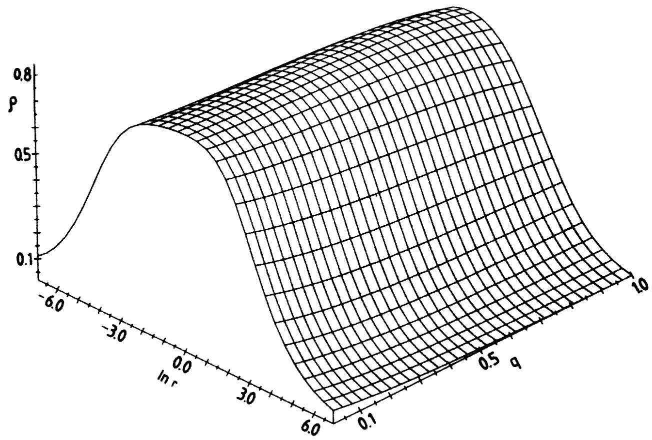

FIG. 5. Reflectivity surface as a function of the pump ratio r and the probe ratio $q=I_{40}/(I_{10}+I_{2d})$. The probe ratio does not affect saturation of the reflectivity appreciably. Here $n=10$ cm^{-1} and $m=2$ cm^{-1}.

In summary, we presented a method for the exact solution of the equal strength multigrating 4WM operation in photorefractive crystals. Such an operation is advantageous for small couplings (nd up to 1) and equal pumps, but it should be avoided for strong couplings ($nd \simeq 10$), since it does not allow signal amplification. In order to achieve reflectivities larger than 1 it is better to operate in the single grating unequal pumps regime. Our solution also allows the treatment of the opposite strength $n_R=-n_T$ operation, but this situation should also be avoided, since it makes the equation for I_3 homogeneous. Because of the boundary conditions the phase conjugate signal is then absent.

The author acknowledges support from the Alexander von Humboldt Foundation.

[1]Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley, New York, 1984), Chap. 14.

[2]*Optical Phase Conjugation,* edited by R. A. Fisher (Academic, New York, 1983); B. Zeldovich, N. Pilipetski, and V. Shkunov, *Principles of Phase Conjugation* (Springer, Berlin, 1985).

[3]A. Yariv, IEEE J. Quantum Electron. **QE-14**, 650 (1978); P. Günter, Phys. Rep. **93**, 199 (1982); M. Cronin-Golomb, B. Fisher, J. O. White, and A. Yariv, IEEE J. Quantum Electron. **QE-20**, 12 (1984).

[4]C. R. Giuliano, Phys. Today **34** (4), 27 (1981).

[5]A. Yariv and D. M. Pepper, Opt. Lett. **1**, 16 (1977); A. Yariv, Opt. Commun. **25**, 23 (1978); N. V. Kukhtarev, V. B. Markov, S. G. Odulov, M. S. Soskin, and V. L. Vinetski, Ferroelectrics **22**, 949 (1979).

[6]M. Cronin-Golomb, B. Fisher, J. O. White, and A. Yariv, Opt. Lett. **7**, 313 (1982).

[7]M. R. Belić, Phys. Rev. A **31**, 3169 (1985); M. R. Belić and M. Lax, Opt. Commun. **56**, 197 (1985).

[8]W. Krolikowski, Opt. Commun. **60**, 319 (1986).