A GENERALIZED NONLINEAR SCHRÖDINGER EQUATION AND THE MOTION OF INHOMOGENEOUS VORTEX FILAMENTS IN A FLUID

Milivoj R. BELIĆ
Institute of Physics, POB 57, 11001 Belgrade, Yugoslavia

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A generalization of the nonlinear Schrodinger equation, arising in the analysis of the motion of inhomogeneous vortex filaments in a fluid is treated by the method of twisted curves and an extended AKNS scheme.

In a recent series of articles [1,2] Balakrishnan has studied the generalized nonlinear Schrödinger equation (GNLSE):

$$\mathrm{i}\dot{u} + (fu)'' + Ru = 0, \quad R \equiv \int^{x} |u|(f|u|)' , \tag{1}$$

where dot denotes the temporal derivative, prime denotes the spatial derivative, and $u(x, t)$ is assumed complex. In ref. [1] an extended AKNS scheme is deviced and applied to the solution of GNLSE, in ref. [2] this equation is connected with the following spin equation:

$$\dot{\hat{s}} = f'\hat{s} \times \hat{s}' + f\hat{s} \times \hat{s}'' , \tag{2}$$

where $\hat{s} = \hat{s}(x, t)$ is a classical unit spin vector. The hamiltonian density giving rise to such an equation is of the form:

$$\mathcal{H} = \tfrac{1}{2} f\hat{s}' \cdot \hat{s}' + \tfrac{1}{2} \int^{x} f'\hat{s}' \cdot \hat{s}' . \tag{3}$$

Another physical problem where GNLSE naturally arises is the motion of inhomogeneous vortex filaments, when the circulation of a vortex $\boldsymbol{\alpha}(x, t)$,

$$\Gamma \equiv \int_C \dot{\boldsymbol{\alpha}} \cdot \mathrm{d}l = \int_A (\nabla \times \dot{\boldsymbol{\alpha}}) \cdot \mathrm{d}A , \tag{4}$$

is assumed to be an arbitrary function $\Gamma = \Gamma(x, t)$; C encircles the vortex, and A is the area enclosed by C. This situation seems realistic when viscosity and diffusion around the vortex is taken into account. The case when Γ is constant, as analysed in refs. [3,4] leads to the ordinary nonlinear Schrödinger equation.

A Frenet trihedron $\hat{t}, \hat{n}, \hat{b}$ is assigned to the space curve $\boldsymbol{\alpha}(x, t)$ representing an isolated vortex, which is swept about in some consistent manner due to the fluid motion. We assume the curve $\boldsymbol{\alpha}$ to be parametrized by its own arc-length, so that $\boldsymbol{\alpha}' = \hat{t}$. On the other hand, its velocity is approximately [4]

$$\dot{\boldsymbol{\alpha}} = \eta(x, t)\kappa\hat{b} , \tag{5}$$

where $\kappa(x, t)$ is the curvature of $\boldsymbol{\alpha}$, and η is a functional of Γ. The spatial and the temporal derivative of the trihedron is given by [3,4]

$$\hat{t}' = \boldsymbol{d} \times \hat{t}, \quad \dot{\hat{t}} = \boldsymbol{\omega} \times \hat{t} , \tag{6}$$

and similarly for the normal $\hat{n}$ and the binormal $\hat{b}$. In (6) $\boldsymbol{d} = \tau\hat{t} + \kappa\hat{b}$ is the Darboux vector, τ is the torsion of $\boldsymbol{\alpha}$, and $\boldsymbol{\omega} = \omega_1\hat{t} + \omega_2\hat{n} + \omega_3\hat{b}$ is the angular velocity of the trihedron. The integrability condition for the trihedron $\hat{t}_{xt} = \hat{t}_{tx}$ etc. leads to the system 3 equations:

$$\dot{\kappa} = \omega_3' + \tau\omega_2 , \quad \dot{\tau} = \omega_1' - \kappa\omega_2, \quad \omega_2' = \tau\omega_3 - \kappa\omega_1 , \tag{7}$$

which, on the one hand contains many soliton equations [3,4] and on the other is connected with the ZS-AKNS two-component scattering problem [5,6]. The integrability condition for $\boldsymbol{\alpha}$ leads to

$$-\eta\kappa\tau\hat{n} + (\eta\kappa)'\hat{b} = \omega_3\hat{n} - \omega_2\hat{b} . \tag{8}$$

With this choice for ω_2 and ω_3, the last equation of the system (7) provides

$$\omega_1 = (\eta\kappa)''/\kappa - \eta\tau^2 , \tag{9a}$$

while the remaining two equations read:

$$\dot{\kappa} + (\eta\kappa\tau)' + (\eta\kappa)'\tau = 0 , \tag{9b}$$

$$\dot{\tau} - [(\eta\kappa)''/\kappa - \eta\tau^2]' - \kappa(\eta\kappa)' = 0 . \tag{9c}$$

However, if $u = \kappa e^{i\sigma}$ with $\sigma' = \tau$ and $\eta = f$ is assumed, these two equations follow exactly when u is substituted into the GNLSE.

In order to treat GNLSE by the inverse scattering procedure, Balakrishnan presented in ref. [1] and extension of the ZS-AKNS eigenvalue problem, in which the eigenvalue is allowed to become space and time dependent. As a consequence, a nonlinear evolution equation for the eigenvalue $\zeta(x, t)$ resulted:

$$i\dot{\zeta} - (f\zeta' + 2if\zeta^2)' = 0 . \tag{10}$$

We note that if $u = q \exp(-2i\int^x \xi)$ is substituted into GNLSE, the following equation is obtained at large distances:

$$i\dot{\xi} + (\tfrac{1}{2}if' + 2f\xi)'' - (f\xi' + 2if\xi^2)' = 0 . \tag{11}$$

So, for f constant, eqs. (10) and (11) are equivalent. Application of the AKNS scheme with time-dependent eigenvalue [7] requires a solution of eq. (10) which separates variables,

$$\zeta(x, t) = g(x)h(t) . \tag{12}$$

While evaluation of h presents no problem, a bit more analysis is needed in the case of g. The remaining part of this letter is therefore devoted firstly to evaluation of the function g, which is crucial in application of the extended AKNS procedure, and secondly to evaluation of the (class of) functions f allowed by such a procedure. So, with the above choice for the solution of eq. (10), two equations follow for g:

$$fg' = \lambda y + \lambda_0 , \quad fg^2 = \mu y + \mu_0 , \tag{13}$$

where $\lambda, \lambda_0, \mu, \mu_0$ is constant, and $y' = g$. Evidently these equations can hold only for certain functions $f(x)$. Considering (13) as a system of equations for g and g' we obtain:

$$g = -\mu f'/d , \quad g' = \mu(f'' + 2\lambda)/d , \tag{14}$$

where

$$d = 2f(f'' + 2\lambda) - (f')^2 . \tag{15}$$

The allowed f's come from the following equation:

$$|d| = C_1(f')^2 \exp(\textstyle\int^x 2\lambda/f') , \tag{16}$$

with $C_1 > 0$ a constant. In general this equation is hard to handle. Only when $\lambda = 0$ we easily recover the result quoted in ref. [1]. A way around is to first find g. To this end we eliminate f between the two equations in (13), and considering g as a function of y, obtain:

$$g = C_2 Y^{\Lambda_0} e^{\Lambda Y} , \tag{17}$$

with $Y = \mu y + \mu_0$, $\Lambda_0 = (\lambda_0\mu - \lambda\mu_0)/\mu^2$, $\Lambda = \lambda/\mu^2$, and C_2 a constant. Thus f is known as a function of y,

$$f = C_3 Y^{1-2\Lambda_0} e^{-2\Lambda Y} , \tag{18}$$

and the actual dependence $y(x)$ is given implicitly through

$$\mu C_2 \Lambda^{1-\Lambda_0}(x - x_0) = \gamma(1 - \Lambda_0, \Lambda Y) , \tag{19}$$

where x_0 is the root of Y, and $\gamma(a, z)$ is the incomplete gamma function [8]:

$$\gamma(a, z) \equiv \int_0^z e^{-t} t^{a-1}\, dt . \tag{20}$$

The case when $\mu = 0$ is treated in a similar manner.

Once g and h are found, an analysis of the direct and inverse scattering problem, as given for example in refs. [1,4,7] leads to the soliton-like solutions of the generalized nonlinear Schrödinger equation. This analysis will not be repeated here.

References

[1] R. Balakrishnan, Phys. Lett. 92A (1982) 243.
[2] R. Balakrishnan, J. Phys. C15 (1982) L1305.
[3] M. Lakshmanan, J. Math. Phys. 20 (1979) 1667.
[4] J.L. Lamb Jr., Elements of soliton theory (Wiley, New York, 1980) ch. 7.
[5] V.E. Zakharov and A.B. Shabat, Sov. Phys. JETP 34 (1972) 62.
[6] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Stud. Appl. Math. 53 (1974) 249.
[7] M.R. Gupta, Phys. Lett. 72A (1979) 420.
[8] M. Abramowitz and J. Stegun, Handbook of mathematical functions (Dover, New York, 1970) ch. 6.