# Symmetries of two-wave mixing in photorefractive crystals

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We consider symmetries of two-wave mixing equations in photorefractive crystals, using group theoretical methods. Symmetry groups for both the equations and conserved quantities are determined, and the corresponding generators are written explicitly. Results obtained for plane-parallel polarized two-wave mixing are used to introduce the method for solution and the form of solutions for cross-polarized two-wave mixing.

Two-wave mixing (2WM) in various nonlinear media is the basic wave mixing process in nonlinear optics.[1] It is the process by which the writing or reading of holograms proceeds or by which the coherent transfer of energy from one laser beam to the other is achieved. Optical phase conjugation[2] by four-wave mixing can be thought of as a composition of a pair of 2WM processes: a two-wave write-in and a simultaneous two-wave readout of volume holograms.

We present an elementary symmetry analysis of slowly varying envelope wave equations describing steady-state 2WM in photorefractive crystals. The symmetries of conserved quantities as well as of the equations are established. The method is introduced by treating the standard 2WM with parallel polarization, and then it is applied to the (hitherto[1] considered unsolved) 2WM with crossed polarizations.

The processes of interest are depicted in Figs. 1 and 2. Figure 1 displays 2WM of two plane-polarized laser beams, $A_1$ and $A_2$, in a photorefractive crystal. This process is described by the following set of wave equations:

$$IA_1' = \Gamma A_1 I_2 - \alpha I A_1, \qquad IA_2' = -\Gamma^* A_2 I_1 - \alpha I A_2 \tag{1}$$

for the transmission geometry (TG) and

$$IA_1' = \Gamma A_1 I_2 - \alpha I A_1, \qquad IA_2' = \Gamma^* A_2 I_1 + \alpha I A_2 \tag{2}$$

for the reflection geometry (RG). $I = I_1 + I_2$ is the total intensity, $\Gamma$ is the wave coupling constant (complex in general but real in photorefractive materials), and $\alpha$ is the linear absorption coefficient. The prime denotes a spatial derivative along the propagation direction, and the asterisk stands for complex conjugation.

Figure 2 depicts 2WM with crossed polarizations in cubic crystals with point symmetry $\bar{4}3m$. This process is described by the equations[1,3]

$$IA_s' = -\Gamma B_p Q - \alpha I A_s, \qquad IA_p' = -\Gamma B_s Q - \alpha I A_p, \tag{3a}$$

$$IB_s' = \Gamma^* A_p Q^* - \alpha I B_s, \qquad IB_p' = \Gamma^* A_s Q^* - \alpha I B_p \tag{3b}$$

for the TG and by

$$IA_s' = \Gamma B_p Q + \alpha I A_s, \qquad IA_p' = \Gamma B_s Q + \alpha I A_p, \tag{4a}$$

$$IB_s' = \Gamma^* A_p Q^* - \alpha I B_s, \qquad IB_p' = \Gamma^* A_s Q^* - \alpha I B_p \tag{4b}$$

for the RG. $A_s$, $A_p$ and $B_s$, $B_p$ are the orthogonally polarized components of the two beams impinging upon the crystal, $Q = A_s B_s^* \pm A_p B_p^*$ is the amplitude of the grating induced by the cross-polarized electric field components (+ for the TG; − for the RG), and $I = I_A + I_B$ is still the total intensity. Parallel couplings in Eqs. (3) and (4) are neglected. Such a wave mixing process is possible, for example, in GaAs when the crystal orientation prevents parallel coupling.[3] However, our method can be generalized to other mixing geometries and coupling mechanisms.[4] The theory of cross-polarized 2WM has been developed by Yeh *et al.*[3]

Our program is as follows. Writing Eqs. (1) and (2) in matrix form, we establish their group symmetry. The symmetries of both the equations and conserved quantities are established by finding the appropriate groups and group generators. The knowledge of symmetries facilitates the solution of equations of interest. The method is generalized to Eqs. (3) and (4) and used to pave the road to their solution. In what follows, linear absorption is neglected.

Let us form the fundamental representation

$$|A\rangle = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \tag{5}$$

so that Eqs. (1) and (2) can be written in matrix form:

$$|A\rangle' = \mathbf{m}|A\rangle, \tag{6}$$

where

$$\mathbf{m} = \begin{bmatrix} 0 & \mu \\ \mp\mu^* & 0 \end{bmatrix} \tag{7}$$

and $\mu = \Gamma A_1 A_2^*/I$. The upper sign is for the TG; the lower, for the RG. From quantum mechanics we recall that an equation $|\psi\rangle' = \mathbf{m}_\psi|\psi\rangle$ has an integral of motion $\langle\psi|\mathbf{n}|\phi\rangle$ if there exists a constant matrix $\mathbf{n}$ such that $\mathbf{n}\mathbf{m}_\psi + \mathbf{m}_\phi^\dagger\mathbf{n} = 0$, where a dagger denotes

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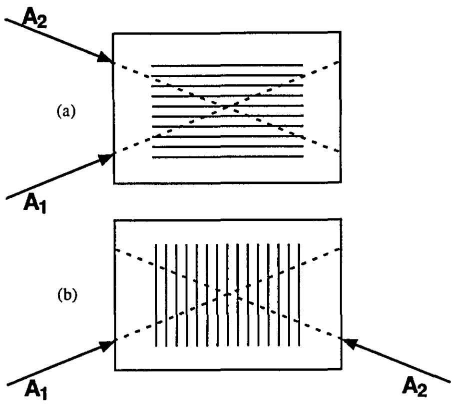


Fig. 1. Two-wave mixing with plane-polarized beams: (a) TG, (b) RG. Parallel lines depict gratings.

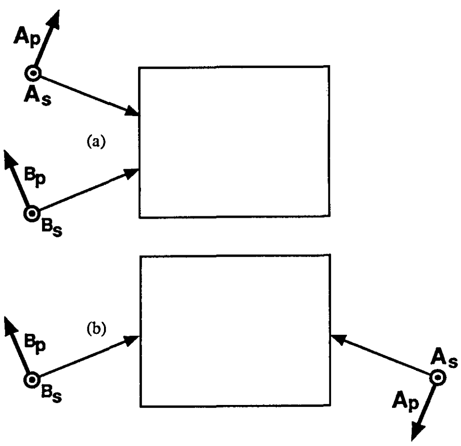


Fig. 2. 2WM with cross-polarized beams: (a) TG, (b) RG.

an adjoint matrix. Here

$$\mathbf{n}_\pm = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \tag{8}$$

and the conserved quantities are

$$q_\pm = I_1 \pm I_2 . \tag{9}$$

They are connected with energy conservation. To simplify the notation, we drop the $\pm$ and just follow the TG.

Conserved quantities are related to the symmetries of the system. We are looking for the symmetries of these constants, i.e., for operators of the form

$$\mathbf{l} = (A_j a_{ij} + A_j{}^* b_{ij})\partial_{A_i} + \text{c.c.} \tag{10}$$

that would satisfy the equation

$$\mathbf{l} q = 0 . \tag{11}$$

Such operators form the set of generators of the symmetry group and constitute the appropriate Lie algebra of the system. In Eq. (10), summation over repeated indices is assumed; c.c. stands for complex conjugation. With a little effort it is found that

$$\mathbf{a} = i\epsilon_0 + i\epsilon_j\sigma_j = \begin{bmatrix} i(\epsilon_0 + \epsilon_3) & i\epsilon_1 + \epsilon_2 \\ i\epsilon_1 - \epsilon_2 & i(\epsilon_0 - \epsilon_3) \end{bmatrix}, \tag{12a}$$

$$\mathbf{b} = (\epsilon_4 + i\epsilon_5)\sigma_2 , \tag{12b}$$

where $\sigma_j$ are the Pauli matrices. This means that a variation of $|A\rangle$ satisfies

$$\delta|A\rangle = i\epsilon_0|A\rangle + i\epsilon_j\sigma_j|A\rangle + (\epsilon_4 + i\epsilon_5)\sigma_2|A^*\rangle . \tag{13}$$

Thus the conserved quantity possesses a six-parameter symmetry $SU(2) \times U(1)^3$. The $U(1)$ symmetries go with the parameters $\epsilon_0$, $\epsilon_4$, and $\epsilon_5$, while the remaining three go with $SU(2)$. The corresponding six generators can be written down explicitly by using Eqs. (10) and (12).

These symmetries do not have to coincide with the symmetries of the equations of motion [Eqs. (1)]. The symmetries of Eqs. (1) are determined by writing an evolution operator for this equation[5]:

$$f = \Gamma A_1 I_2 \partial_{A_1} - \Gamma^* A_2 I_1 \partial_{A_2} + \Gamma^* A_1{}^* I_2 \partial_{A_1}{}^* - \Gamma A_2{}^* I_1 \partial_{A_2}{}^* \tag{14}$$

and by finding all operators $L$ satisfying the equation

$$I[f, L] + L[I]f = 0 , \tag{15}$$

where $[\,,]$ denotes a commutator. Operators $L$ are again sought in the most general linear form [Eq. (10)]. After some algebra, one arrives at the following generators:

$$L_0 = A_1\partial_{A_1} + A_2\partial_{A_2} + \text{c.c.} , \tag{16a}$$

$$L_1 = i(A_1\partial_{A_1} - \text{c.c.}) , \tag{16b}$$

$$L_2 = i(A_2\partial_{A_2} - \text{c.c.}) . \tag{16c}$$

The first generator represents the dilational symmetry; the other two, the freedom in the choice of initial phases of the fields $A_1$ and $A_2$. The total symmetry of the dynamical equations [Eqs. (1)] is thus $R \times U(1)^2$.

The situation with respect to the reflection case changes inasmuch as the $SU(2)$ symmetry group of the integral of motion changes into $SU(1,1)$. This means that, in the presumed solution for $A_1$ and $A_2$, which takes into account the form of the conserved quantity [Eq. (9)], the sin and cos functions in the TG should be replaced with the sinh and cosh functions in the RG. However, this information cannot be used to construct the unknown solutions of the RG in 2WM with crossed polarizations from the known solutions of the TG.[1] Let us consider in some detail the 2WM equations with crossed polarizations.

There are now four field components, which can be organized into two fundamental representations:

$$|\psi_1\rangle = \begin{pmatrix} A_s \\ B_p \end{pmatrix}, \qquad |\psi_2\rangle = \begin{pmatrix} A_p \\ B_s \end{pmatrix}. \tag{17}$$

Equations (3) and (4) can be rewritten as

$$|\psi_1\rangle' = \mathbf{m}|\psi_1\rangle, \qquad |\psi_2\rangle' = \mathbf{m}|\psi_2\rangle, \tag{18}$$

where $\mathbf{m}$ has the same form as above, with $\mu = \mp\Gamma Q/I$. There are now two $\mathbf{n}$ matrices:

$$\mathbf{n}_\pm = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \qquad \mathbf{n} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{19}$$

Written in this form, 2WM with crossed polarizations resembles a 4WM process. Therefore we apply the methodology developed earlier[4] for 4WM processes. The conserved quantities are

$$\langle\psi_1|\mathbf{n}_+|\psi_1\rangle = A_sA_s^* + B_pB_p^* = c_1, \tag{20a}$$

$$\langle\psi_2|\mathbf{n}_+|\psi_2\rangle = A_pA_p^* + B_sB_s^* = c_2, \tag{20b}$$

$$\langle\psi_2|\mathbf{n}_+|\psi_1\rangle = A_sA_p^* + B_s^*B_p = c_3, \tag{20c}$$

$$\langle\psi_2^*|\mathbf{n}|\psi_1\rangle = A_pB_p - A_sB_s = c_4 \tag{20d}$$

for the TG and

$$\langle\psi_1|\mathbf{n}_-|\psi_1\rangle = A_sA_s^* - B_pB_p^* = b_1, \tag{21a}$$

$$\langle\psi_2|\mathbf{n}_-|\psi_2\rangle = A_pA_p^* - B_sB_s^* = b_2, \tag{21b}$$

$$\langle\psi_2|\mathbf{n}_-|\psi_1\rangle = A_sA_p^* - B_s^*B_p = b_3, \tag{21c}$$

$$\langle\psi_2^*|\mathbf{n}|\psi_1\rangle = A_pB_p - A_sB_s = b_4 \tag{21d}$$

for the RG. Similar relations are derived in Ref. 3. Another quadratic conserved quantity is found for the RG when $\Gamma$ is real (which is the case here):

$$\bar{b}_3 = A_sA_p^* - B_sB_p^*. \tag{21e}$$

A formally analogous quantity for the TG

$$\bar{c}_3 = A_sA_p^* - B_sB_p^*, \tag{20e}$$

is constant when $\Gamma$ is imaginary (which is of no interest here). Also, not all the conserved quantities are independent. It is easy to check that

$$|c_3|^2 + |c_4|^2 = c_1c_2, \tag{22a}$$

$$|b_3|^2 - |b_4|^2 = b_1b_2. \tag{22b}$$

Further, there exist higher-order (quartic) conserved quantities, notably for the TG:

$$(I_A - I_B)^2 + 4|R|^2 = \text{const.} \tag{23a}$$

and for the RG:

$$(I_A + I_B)^2 - 4|R|^2 = \text{const.}, \tag{23b}$$

where $I_A = |A_s|^2 + |A_p|^2$, $I_b = |B_s|^2 + |B_p|^2$, $R = A_sB_p^* + B_s^*A_p$. For real $\Gamma$ and RG (or imaginary $\Gamma$ and TG) the modulus of the grating amplitude $|Q|^2$ is also constant. Unlike the case of true 4WM,[4] these quantities cannot be used for an obvious integration of Eqs. (3) and (4).

A brute-force method based on Eq. (15) reveals that there are four independent generators for Eqs. (3) and (4). There are two additional generators for the RG when $\Gamma$ is real and an additional two for the TG when $\Gamma$ is imaginary. One can similarly derive the symmetries of conserved quantities. It suffices to note that the nontrivial symmetry (sub)-groups remain as $SU(2)$ and $SU(1,1)$. Using this symmetry information, one can solve Eqs. (3) and (4) in quadratures if one assumes a solution of the form

$$A_s = c_1^{1/2} \cos \alpha_1 \exp(i\beta_1), \tag{24a}$$

$$A_p = c_2^{1/2} \cos \alpha_2 \exp(i\beta_2), \tag{24b}$$

$$B_s = c_2^{1/2} \sin \alpha_2 \exp(i\gamma_2), \tag{24c}$$

$$B_p = c_1^{1/2} \sin \alpha_1 \exp(i\gamma_1) \tag{24d}$$

for the TG and

$$A_s = b_1^{1/2} \cosh \alpha_1 \exp(i\beta_1), \tag{25a}$$

$$A_p = b_2^{1/2} \cosh \alpha_2 \exp(i\beta_2), \tag{25b}$$

$$B_s = b_2^{1/2} \sinh \alpha_2 \exp(i\gamma_2), \tag{25c}$$

$$B_p = b_1^{1/2} \sinh \alpha_1 \exp(i\gamma_1) \tag{25d}$$

for the RG. The space of new variables $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ can be visualized as a direct product of two 3-spheres (TG) or 3-hyperboloids (RG). Not all these variables are independent. With the use of conserved quantities, and by solving one of Eqs. (3) or (4) explicitly, a solution in quadratures is obtained. It turns out that all variables in Eqs. (24) and (25) can be expressed in terms of a combination $u = \alpha_1 + \alpha_2$, and the explicitly solved equation yields a relation $z = f(u)$.

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