Degenerate-four-wave mixing as a Sturm–Liouville problem

M. R. Belić

Max-Planck-Institut für Quantenoptik, D-8046 Garching, Federal Republic of Germany

Received July 25, 1986; accepted October 30, 1986

The problem of steady-state degenerate holographic four-wave mixing in transmission geometry in nonlinear dynamic media is reduced to a Sturm–Liouville or a one-dimensional quantum-mechanical scattering problem, which is treated exactly. Linear absorption in the medium is accounted for, and pump depletion is allowed. No restrictions are placed on the spatial phase mismatch between light-interference fringes and refractive-index gratings. Energy and phase transfer are considered simultaneously.

The production of phase-conjugated or time-reversed wave fronts in photorefractive crystals and other dynamic (real-time holographic) media has attracted much experimental and theoretical attention for its great applicative potential.[1] The building of phase-conjugate mirrors and other optical elements has had a great effect on optical signal processing (transmission through fibers), adaptive optics (correction of aberrations), and real-time holography, to mention few areas of applications. The attainment of strong and effective wave coupling in such media has significantly lowered power requirements on the laser pump beams. Consequently the effects of pump depletion and absorption in the medium can no longer be ignored, and an urgent need has been created for solutions of the theories of wave mixing that include both of these effects.[2]

Prominent theories or models of degenerate four-wave mixing (FWM) in dynamic media are due to a Russian school from Kiev[3] and to an American group around Yariv.[2,4] The Russians are using mostly intensities and the relative phase as the relevant variables, while Americans are using complex wave amplitudes. The theories are essentially equivalent and have been treated so far in various degrees of approximation.[1–4] The attempts to treat them exactly in their general form (depletion and absorption, energy and phase, different geometries) have thus far proven futile.

In this Letter we undertake two tasks. First, we solve exactly the theory of steady-state holographic FWM in transmission geometry in its general form, as presented in Ref. 2. Thus we allow for pump depletion and absorption in the medium and place no constraints on the spatial phase difference between refractive-index gratings and the light-interference pattern. We not only consider energy transfer (i.e., variations in the beam intensities for exact phase conjugation) but retain arbitrary phase variations in the fields as well. Second, we connect this model with a one-dimensional quantum-mechanical scattering or Sturm–Liouville problem (also exactly solvable) and bring powerful methods of quantum theory to the rescue.

Our starting point is the following set of stationary wave equations in the slowly varying amplitude approximation for the pump beams A_1 and A_2, the signal A_3, and the phase-conjugate output A_4, all plane waves[2]:

$$A_1' = \frac{\Gamma}{I}(A_1A_3^* + A_2^*A_4)A_3 - \alpha A_1, \tag{1a}$$

$$A_2^{*\prime} = \frac{\Gamma}{I}(A_1A_3^* + A_2^*A_4)A_4^* + \alpha A_2^*, \tag{1b}$$

$$A_3^{*\prime} = -\frac{\Gamma}{I}(A_1A_3^* + A_2^*A_4)A_1^* - \alpha A_3^*, \tag{1c}$$

$$A_4' = -\frac{\Gamma}{I}(A_1A_3^* + A_2^*A_4)A_2 + \alpha A_4, \tag{1d}$$

where $\Gamma = \Gamma_0 \exp[i(\pi/2 - \phi)]$ is the complex coupling constant (ϕ is the angle between index and interference gratings), $I = |A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2$ is the total intensity, and α is the absorption constant. The prime denotes the derivative in the propagation z direction. We thus consider a transmission geometry of the standard FWM setup: two counterpropagating laser pumps illuminate a nonlinear medium situated between the planes $z = 0$ and $z = d$. From the left (and tilted for a small angle) comes the signal A_3, and out of the medium, in the same direction, goes the output A_4. For simplicity, all fields are assumed to be of the same polarization. One is interested in the functional dependence of the fields $A_j(z)$ in the region $0 \leq z \leq d$, i.e., in the solution of the system [Eqs. (1)] when boundary conditions are applied at the exit planes, that is, when A_{10}, A_{30}, A_{2d}, and $A_{4d} = 0$ are known. From these, the quantities of experimental interest (e.g., the reflectivity $\rho = A_{40}/A_{30}^*$) are easily constructed.

The analysis of Eqs. (1) proceeds as follows. First, by combining them, four of the Manley–Rowe relations are obtained:

$$A_1A_2 + A_3A_4 = c_1, \tag{2a}$$

$$A_2A_3^* - A_1^*A_4 = c_2, \tag{2b}$$

$$|A_1|^2 + |A_3|^2 = u_1, \tag{2c}$$

$$|A_2|^2 + |A_4|^2 = u_2, \tag{2d}$$

where c_1 and c_2 are constant and $u_1 = (I_{10} + I_{30})\exp(-2\alpha z)$ and $u_2 = I_{2d} \exp[2\alpha(z - d)]$ are given

0146-9592/87/020105-03$2.00/0

functions of z. By using these relations (which, by the way, are not all independent; for example, $u_1u_2 = |c_1|^2 + |c_2|^2$), and in the style of Ref. 2, two Riccati equations of the form

$$y' = \frac{\Gamma}{u_1 + u_2}[\pm c_1 + (u_1 - u_2)y \mp c_1{}^*y^2] \mp 2\alpha y \quad (3)$$

are obtained for the quantities $A_1/A_2{}^*$ (the upper sign) and $A_4/A_3{}^*$ (the lower sign). A transformation of the dependent $y \rightarrow (\pm v'/\Gamma c_1{}^* v)(u_1 + u_2)$ and independent variables $2\alpha z - \mu \rightarrow \zeta$, where $\tanh\mu = [I_{10} + I_{30} - I_{2d}\exp(-2\alpha d)]/[I_{10} + I_{30} + I_{2d}\exp(-2\alpha d)]$, turns the Riccati equation into a second-order linear equation:

$$v'' + [(1 + 2\delta)\tanh\zeta \pm 1]v' - \frac{\delta^2|c_1|^2}{b^2}\operatorname{sech}^2\zeta\, v = 0, \quad (4)$$

with $\delta = (\Gamma/4\alpha)$ and $b^2 = u_1u_2$. The prime here denotes differentiation with respect to ζ. Finally, another change of the independent variable, $1 - \tanh\zeta \rightarrow 2\xi$, brings the equation for v into the hypergeometric equation

$$\xi(\xi - 1)v'' + [(\tfrac{1}{2} - \delta)2\xi - \gamma]v' + \frac{\delta^2|c_1|^2}{b^2}v = 0, \quad (5)$$

where γ equals $-\delta$ for $A_1/A_2{}^*$ and $1 - \delta$ for $A_4/A_3{}^*$. In such a manner the problem is reduced to an exercise in special functions theory. A few specific points, however, can simplify the analysis.

First, the last coordinate transformation reduced the range of the independent variable to the interval (0, 1). Therefore one need consider only the solutions of the hypergeometric equation around regular singular points at $\xi = 0$ and/or $\xi = 1$.

Second, solutions to Eq. (5) for $A_1/A_2{}^*$ and $A_4/A_3{}^*$ are contiguous.[5] General analysis then can benefit from Gauss relations and need not be repeated for both cases.

Finally, when Eq. (4) is put into its normal form (which, as we shall see, turns out to be a simple Schrödinger equation), the whole problem is transformed into a convenient language of one-dimensional quantum mechanics. The two problems can be analyzed in parallel (without, however pushing this formal analogy too far).

Two pairs of linearly independent solutions of Eq. (5) can be picked up from Kummer's list[5] of 24 solutions to the hypergeometric equation, and for $A_4/A_3{}^*$ they are of the form

$$v_1(\xi) = F(-\delta + \epsilon, -\delta - \epsilon; 1 - \delta; \xi)$$
$$= (1 - \xi)^{1+\delta}F(1 - \epsilon, 1 + \epsilon; 1 - \delta; \xi), \quad (6a)$$

$$v_2(\xi) = \xi^\delta F(\epsilon, -\epsilon; 1 + \delta; \xi) \quad (6b)$$

around $\xi = 0$ and

$$v_1(\xi) = F(-\delta + \epsilon, -\delta - \epsilon; -\delta; 1 - \xi)$$
$$= \xi^\delta F(-\epsilon, \epsilon; -\delta; 1 - \xi), \quad (7a)$$

$$v_2(\xi) = (1 - \xi)^{1+\delta}F(1 + \epsilon, 1 - \epsilon; 2 + \delta; 1 - \xi) \quad (7b)$$

around $\xi = 1$ (and for δ noninteger). Here ϵ stands for $\delta|c_2|/b$, and F denotes the standard Gauss hypergeometric function. The pairs of fundamental solutions for $A_1/A_2{}^*$ are the same, except that ξ is replaced by $1 - \xi$ everywhere. In other words, if the solutions of the hypergeometric equation about $\xi = 0$ are chosen as the basic set for $A_4/A_3{}^*$, then the same functions (with ξ replaced by $1 - \xi$) will give the basic solutions for $A_1/A_2{}^*$ about $\xi = 1$. The only unknown parameter $|c_1|$ that figures in the fundamental solutions is evaluated below from the boundary conditions.

Continuing along similar lines, by a further change of the dependent variable, $\psi = v\cosh^{\delta+1/2}\xi\exp(\pm\xi/2)$, Eq. (4) is brought to its normal form $\psi'' + T\psi = 0$, where

$$T = [-(\tfrac{1}{2} + \delta)^2 - \tfrac{1}{4}]$$
$$- [\pm(\tfrac{1}{2} + \delta)\tanh\xi + (\tfrac{1}{4} - \epsilon^2)\operatorname{sech}^2\xi]. \quad (8)$$

Thus for δ real (photorefractive media) a Schrödinger equation is obtained, with the potential and possible states depicted in Fig. 1. More precisely, a Sturm–Liouville boundary-value problem of the first kind is obtained, since boundary conditions are given at finite values of ξ. Furthermore, the accessibility of various displayed states places stringent conditions on our parameters, which physically may not be plausible or warranted. In fact, a realizable quantum-mechanical analogy [with negative energy in Eq. (8)] would favor bound states. A detailed account will be published elsewhere.

With an identification[6] $\tanh\mu = -(1/2 + \delta)/(1/2 - 2\epsilon^2)$ and $V_0 = (1/2 + \delta)^2/(1 - 4\epsilon^2) - 1/4 + \epsilon^2$, the potential gets the form

$$V = V_0\cosh^2\mu(\tanh\mu \pm \tanh\xi)^2 \quad (9)$$

up to a constant, and the scattering problem of this potential was considered in detail by Morse and Feshbach.[7]

Going backward, from the solutions of the hypergeometric equation or the Schrödinger equation the expressions for $y_1 = A_1/A_2{}^*$ and $y_2 = A_4/A_3{}^*$ are found:

$$y_1 = \frac{I}{\Gamma c_1{}^*} \times \frac{\Gamma|c_1|^2(v_{1d}v_2' - v_{2d}v_1') - I_dI_{2d}(v_{1d}'v_2' - v_{2d}'v_1')}{\Gamma|c_1|^2(v_{1d}v_2 - v_{2d}v_1) - I_dI_{2d}(v_{1d}'v_2 - v_{2d}'v_1)}, \quad (10a)$$

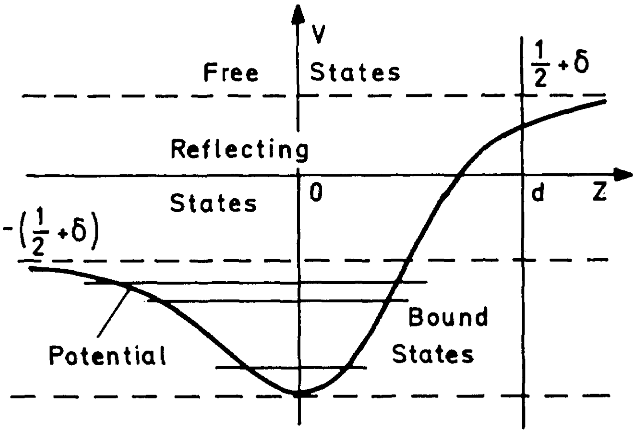

Fig. 1. Typical shape of the scattering potential [Eq. (8) or (9)] for the Schrödinger equation of the holographic degenerate FWM. In a realizable quantum-mechanical analogy with negative energy, bound states are preferred.

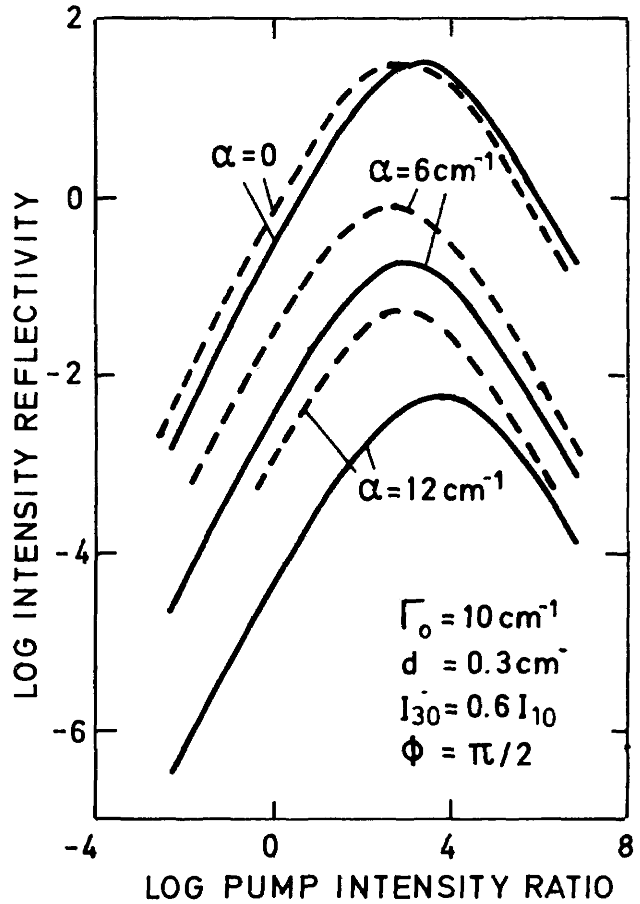

Fig. 2. Intensity reflectivity I_{40}/I_{30} versus pump ratio I_{2d}/I_{10} for various values of the absorption α. Dashed lines represent the undepleted-pumps theory of Ref. 2.

$$y_2 = \frac{I}{\Gamma c_1^*} \frac{v_{1d}'v_2' - v_{2d}'v_1'}{v_{1d}'v_2 - v_{2d}'v_1}, \tag{10b}$$

where in each of the y's the corresponding set of fundamental solutions v_1 and v_2 is supposed to be taken. The differentiation, here with respect to z, is trivial. From these, and from the Manley–Rowe relations, the intensities are found:

$$I_2 = \frac{u_2 - |y_2|^2 u_1}{1 - |y_1 y_2|^2}, \qquad I_3 = \frac{u_1 - |y_1|^2 u_2}{1 - |y_1 y_2|^2}, \tag{11a}$$

$$I_1 = |y_1|^2 I_2, \qquad I_4 = |y_2|^2 I_3. \tag{11b}$$

The only missing ingredient, the value of the constant c_1, is found by applying boundary conditions to y_1 and y_2 at $z = 0$, i.e., by substituting the expression for $\rho = y_{20}$ into the condition for $y_{10} = I_{10}/(c_1^* - \rho^* I_{30})$. In this manner an implicit equation for c_1 is obtained.

Evaluation of the phases also presents no problem. Denoting by z_j the phase factors $\exp(i\phi_j)$ of the various beams, and by θ_1 and θ_2 the (known) phases of y_1 and y_2, the Manley–Rowe relations yield three expressions for z_j:

$$z_1 z_2 = \exp(i\theta_1), \qquad z_3 z_4 = \exp(i\theta_2), \tag{12a}$$

$$c_2 z_2 z_4 = |A_2 A_3| \exp(i\theta_2) - |A_1 A_4| \exp(i\theta_1), \tag{12b}$$

from which, for example, z_1, z_3, and z_4 can be expressed in terms of z_2. Then z_2 (or ϕ_2) is found by integrating one of the original Eqs. (1):

$$\phi_2 = \phi_{2d} + \Gamma_0 \int_z^d \frac{dz'}{u_1 + u_2} \{J \sin\phi \sin(\theta_1 - \theta_2) + \cos\phi\, [I_4 + J\cos(\theta_2 - \theta_1)]\}, \tag{13}$$

where J denotes $(I_1 I_3 I_4 / I_2)^{1/2}$ and the intensities are given by Eqs. (11).

The effects of absorption are displayed in Fig. 2, in which the intensity reflectivity $|y_{20}|^2$ is plotted as a function of the pump intensity ratio. It is seen that these effects are always deleterious. The theory presented of course contains as a special case the absorptionless theory of Ref. 2. Then the fundamental solutions for A_4/A_3^*, for example, are given with $v_1 = \exp[-\Gamma(\Delta - Q)z/2I]$ and $v_2 = \exp[-\Gamma(\Delta + Q)z/2I]$, where $\Delta = u_2 - u_1$ and $Q = (\Delta^2 + 4|c_1|^2)^{1/2}$, and the remaining analysis applies without any change.

In summary, we have obtained an exact, closed-form solution to the problem of degenerate FWM in transmission geometry with pump depletion and linear absorption accounted for. Arbitrary phase mismatch between the interference fringes and the refractive-index pattern is allowed, and both the intensity and the phase variations of the beams are considered simultaneously.

The author acknowledges support from the Alexander von Humboldt Foundation. He is also grateful to H. Walther for useful discussions and to the International Centre for Theoretical Physics in Trieste for hospitality during the Spring College in Condensed Matter Physics, where this work was completed. He is also with the Institute of Physics, P.O. Box 57, 11001 Belgrade, Yugoslavia.

References

1. A comprehensive overview of the field is given in R. A. Fisher, ed., *Optical Phase Conjugation* (Academic, New York, 1983).
2. M. Cronin-Golomb, B. Fisher, J. O. White, and A. Yariv, Opt. Lett. 7, 313 (1982); IEEE J. Quantum Electron. **QE-20,** 12 (1984).
3. N. V. Kukhtarev, V. B. Markov, S. G. Odulov, M. S. Soskin, and V. L. Vinetski, Ferroelectrics **22,** 949, 961 (1979).
4. A. Yariv, IEEE J. Quantum Electron. **QE-14,** 650 (1978); A. Yariv and D. M. Pepper, Opt. Lett. **1,** 16 (1977). For an early attempt to include absorption see R. L. Abrams and R. C. Lind, Opt. Lett. **2,** 94 (1978); **3,** 205 (1978).
5. M. Abramowitz and J. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1953), Chap. 15; A. Erdelyi, ed., *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1, Chap. 2.
6. A note of caution: identity on tanh μ here produces a relation between boundary values and an independent parameter δ.
7. P. M. Morse and H. Feshback, *Mathematical Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), p. 1651.