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Interconnected ring oscillators

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Abstract. Threshold and operation conditions of the interconnected photorefractive ring oscillators are investigated, using the grating action method. Three types of two connected oscillators are considered, the first one consisting of two transmission grating crystals, the second one of two reflection grating crystals, and the third one of one transmission and one reflection grating crystal. It is found that, under similar conditions, the transmission–transmission ring possesses the lowest threshold and attains the highest reflectivity of the three.

Keywords: Four-wave mixing, ring resonators, photorefractive media

1. Introduction

Photorefractive (PR) ring oscillators are interesting for potential applications in optical communications and computing [1–4]. Of special interest are those geometries that produce phase conjugate (PC) beams of mutually incoherent input beams. The double PC mirror system [5] was, for a long time, considered to be the favourite geometry. Mamaev and Zozulya [6] introduced three novel types of interconnected ring geometries that allow for such phase conjugation. It seemed appropriate to investigate their threshold and operation conditions.

The geometries of interest are presented in figure 1. They involve two PR crystals and two incident beams. After passing through the crystal, each beam is directed (with the help of external mirrors) to intersect with the other beam in the other crystal. In this manner two four-wave mixing (4WM) regions are formed in the crystals, and depending on the type of diffraction gratings that the incident and scattered waves produce, three kinds of interconnected rings are possible. One kind is when the transmission gratings (TGs) are prevalent in both regions, the other kind is when the reflection gratings (RGs) are prevalent, and the third kind is when one grating is a TG and the other is a RG. Even though all three kinds of ring are realized experimentally in [6], only results concerning the TG–TG ring are reported there.

We consider all three cases of interconnected rings theoretically, using the method of grating action [7]. We investigate the threshold and operation conditions, and compare the strengths and weaknesses of the different geometries. We find that the TG–TG ring is the preferable geometry, possessing the lowest threshold and the highest PC reflectivity of the three.

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In section 2 of this paper the method is introduced, and section 3 contains some examples. The following three sections contain the discussion of each individual geometry, and section 7 summarizes the conclusions.

2. Method

General methods for solution of 4WM equations in PR media have been developed by Cronin-Colomb *et al* [1] and Zozulya and Tikhonchuk [8]. Both methods lead to complicated expressions. We use the grating action method, which offers simpler expressions, but is applicable only to plane-wave, degenerate, and steady-state cases of 4WM. The coupling constants are real then and the phases of grating amplitudes are constant. In reality, these conditions are not very restrictive, and are met in most of the PC situations. Specifically, they are convenient for the situations where the threshold and operation conditions of PR oscillators are investigated. We introduce the method by analysing the simple TG and RG rings, which will be instructive for the later more complicated interconnected cases.

Considering an isolated 4WM interaction region (IR) in a TG, the output waves are connected with the input waves through one quantity, the grating action u [4]

$$A_{1d} = A_{10}\cos u - A_{40}\sin u, \qquad A_{4d} = A_{10}\sin u + A_{40}\cos u, \tag{1a}$$

$$A_{20} = A_{3d}\sin u + A_{2d}\cos u, \qquad A_{30} = A_{3d}\cos u - A_{2d}\sin u, \tag{1b}$$

where the subscripts 0 and d denote the two entrance/exit faces of the IR, and u is defined as

$$u = \frac{1}{d}\int_0^d \frac{\Gamma|Q|}{I}\,dz, \tag{2}$$

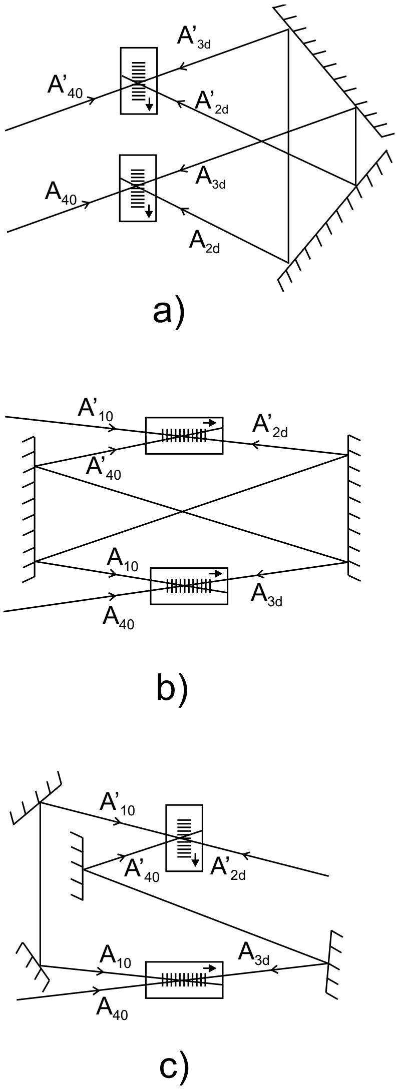

Figure 1. Interconnected rings. (*a*) TG–TG ring, (*b*) RG–RG ring, (*c*) TG–RG ring.

where Γ is the coupling strength of wave mixing (amplitude coupling constant times the thickness d of the IR), $Q = A_1\overline{A}_4 + \overline{A}_2A_3$ is the amplitude of the grating, and $I = |A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2$ is the total intensity. The grating action is found from the expression

$$(A_{10}\overline{A}_{40} + \overline{A}_{2d}A_{3d} + \text{c.c.})\cot u + I_{10} - I_{2d} + I_{3d} - I_{40} = aI_0 \coth\left(\frac{a\Gamma}{2}\right), \tag{3}$$

which involves only the input waves. The overbar stands for complex conjugation. The ‘order’ parameter a is found from the conserved quantity

$$a^2I^2 = 4|Q|^2 + P^2, \tag{4}$$

where $P = I_1 + I_2 - I_3 - I_4$ is connected with the Poynting flow through the crystal. a is called the order parameter because its minimal value signifies the onset of oscillation.

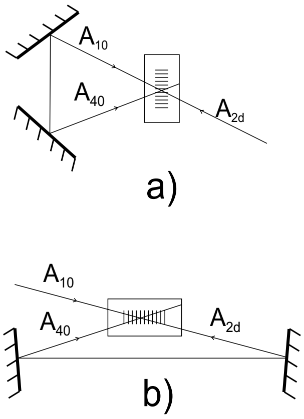

Figure 2. Simple rings. (*a*) TG ring, (*b*) RG ring.

Similar expressions hold for the 4WM process through the RG. The output and input waves are connected via [4]

$$\begin{aligned} A_{1d} &= A_{10}\,\text{sech}\,u + A_{3d}\tanh u, \\ A_{4d} &= A_{40}\,\text{sech}\,u + A_{2d}\tanh u, \end{aligned} \tag{5a}$$

$$\begin{aligned} A_{20} &= A_{2d}\,\text{sech}\,u - A_{40}\tanh u, \\ A_{30} &= A_{3d}\,\text{sech}\,u - A_{10}\tanh u, \end{aligned} \tag{5b}$$

and the same equation (2) defines the grating action, with the difference that $Q = A_1\overline{A}_3 + \overline{A}_2A_4$ now stands for the amplitude of the RG. The practical expression giving u is

$$(A_{10}\overline{A}_{3d} + \overline{A}_{2d}A_{40} + \text{c.c.})\,\text{cosech}\,u + I_{2d} + I_{3d} - I_{10} - I_{40} = (I_{10} + I_{40} + I_{2d} + I_{3d})\coth\left(\frac{\Gamma}{2}\right). \tag{6}$$

There exists no order parameter in the RG, or formally $a = 1$. The RG expressions can be formally obtained from the TG expressions by a symmetry transformation:

$$\{\sin u, \cos u\}_{\text{TG}} \longleftrightarrow \{-\tanh u, \text{sech}\,u\}_{\text{RG}}, \tag{7a}$$

provided that beams 3 and 4 are switched at the same time:

$$\{I_{40}, I_{4d}\}_{\text{TG}} \longleftrightarrow \{I_{3d}, I_{30}\}_{\text{RG}}. \tag{7b}$$

The quantities of practical interest are the reflectivities and transmissivities of the 4WM process:

$$R = I_{\text{reflected}}/I_{\text{input}}, \qquad T = I_{\text{output}}/I_{\text{input}}. \tag{8}$$

Real devices or PC mirrors are formed by connecting the output beams of one or more IR with the input beams, using external mirrors or total internal reflections within the crystal. A number of such devices are in use [1,2,5,9]. For the purpose of presenting the method, and for later convenience, we analyse two such devices, the TG and RG rings.

3. TG and RG rings

The simple TG and RG rings are depicted in figure 2. The ring passive PC mirror in TG was first analysed in [1]. By the grating action method it has been analysed in [10]. The RG self-pumped ring mirror was investigated by Dyakov *et al* [11] using the method of [6]. We apply the grating action method. In the TG geometry the external mirrors provide additional conditions on the input beams $A_{40} = tA_{20}$ and $A_{10} = tA_{30}$, where t is the transmissivity of the feedback loop. When these conditions are used in equations (3) and (4), one obtains the following expressions:

$$1 + a\coth\left(\frac{a\Gamma}{2}\right) = \frac{2|t|^2}{1+|t|^2}(\sin^2 u - \cos^2 u), \qquad (9a)$$

$$\cos(2u) = \frac{1+|t|^2}{2|t|}\sqrt{1-a^2}, \qquad (9b)$$

$$R = 4|t|^2 \sin^2 u \cos^2 u. \qquad (9c)$$

The threshold is found in the limit $u \to 0$,

$$a_{\text{th}}\Gamma_{\text{th}} = \ln\frac{2|t|^2}{1+|t|^2}, \qquad (10a)$$

where the minimum value of a equals

$$a_{\text{th}} = \frac{1-|t|^2}{1+|t|^2}. \qquad (10b)$$

The steady state is determined by solving equations (9a) and (9b) for u and a, given the values of Γ and t. Equation (9c) then supplies the steady-state value of R. The most efficient operation is achieved at the maximum value of R, which occurs at $u_{\text{m}} = -\pi/4$. At that point $a \to 1$, which implies $\Gamma \to -\infty$. This, of course, is not possible. Nonetheless, given the range of Γ that can be achieved in the crystal, the device will operate at the maximum value of the coupling strength available.

The major difference between the TG and RG procedure is the lack of the order parameter in the RG case. The grating action analysis, when applied to the RG ring mirror, leads to the expressions

$$1 + \coth\left(\frac{\Gamma}{2}\right) = \frac{2|t|^2}{1+|t|^2}(\tanh^2 u - \text{sech}^2 u), \qquad (11a)$$

$$R = 4|t|^2 \tanh^2 u \, \text{sech}^2 u. \qquad (11b)$$

The threshold coupling is given by

$$\Gamma_{\text{th}} = \ln\frac{|t|^2}{1+2|t|^2}, \qquad (12)$$

and the reflectivity is maximum ($R = |t|^2$) at the point $u_{\text{m}} = -\sinh^{-1} 1$. This again implies $\Gamma \to -\infty$. Hence, the device will tend to operate at the highest value of Γ available. For such a Γ one solves equation (11a) for the operative value of u, and determines the steady-state value of R. Figure 3 compares the reflectivities of the TG and RG rings.

In the case of devices with one IR, the analysis is simple. For devices with more than one IR, one has to apply the same analysis to each. The complication is that one obtains systems of coupled nonlinear algebraic equations, and to determine the threshold and operation conditions one has to employ Fermat's principle, which requires that the device should operate at the extremum value of the total grating action.

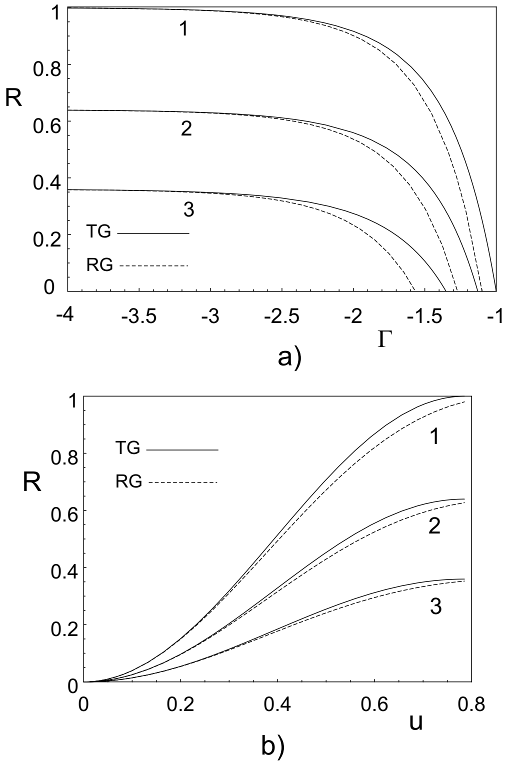

Figure 3. Reflectivities of the simple rings (a) as functions of the coupling strength and (b) as functions of the grating action, for different values of the mirror reflectivities. (1) $|t| = 1$, (2) $|t| = 0.8$, (3) $|t| = 0.6$.

Actually, Fermat's principle requires that the optical path within the device is extremal. The optical path is the line integral over the index of refraction along the propagation path. In a PR crystal, the index of refraction contains the contributions from the bulk n_0 plus the change, $n_1 \sim |\Gamma Q|/I$, coming from the grating [1]. This change is sinusoidally modulated by the grating wavevector K, i.e. it represents the amplitude of the change in the index of refraction. When integrated over the propagation path, it gives the grating action multiplied by the thickness d of the IR†. Hence, the physical meaning of the grating action is that it represents the magnitude of the change in the optical path within the crystal, due to the establishment of 4WM gratings.

Before going on to the details of each individual geometry, we should note some features that are common to all of them. First, owing to the geometry of coupling, there exists only one expression for the transmissivity of each of the devices. This is less obvious for the TG–RG ring, but nonetheless it is true. Second, the relation between the reflectivities of each device is given by

$$R'/|q|^2 = |q|^2 R, \qquad (13)$$

where $|q|^2$ is the ratio of input intensities and R' and R are the reflectivities of the two input beams. Hence, one of the reflectivities can be larger than 1.

† The integral over the modulated change does not average to zero, because $|\Gamma Q|/I$ is not constant. This is one of the differences between PR dynamical gratings and fixed gratings.

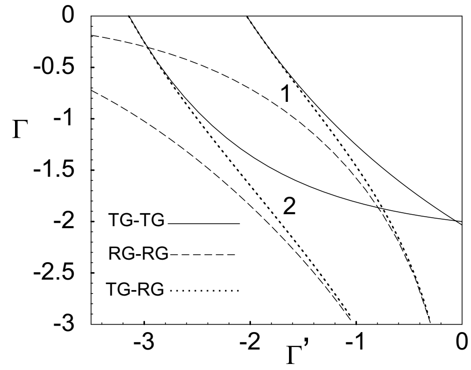

Figure 4. Threshold curves of the interconnected rings for different values of $|t|$ and $|q|$. (1) $|t| = 0.8$, $|q| = 1$; (2) $|t| \approx 0.5$, $|q| = 0.5$.

4. TG–TG ring

Some of the results concerning the TG–TG ring have been reported in [12]. Interconnected rings contain (at least) two IRs. To distinguish between them, we denote one IR by a prime. When equations (3) and (4) are applied to the situation presented in figure 1(*a*), the following relations are obtained:

$$\cos(u + u') = \frac{1 + |tq|^2}{2|tq|}\sqrt{1 - a'^2}, \tag{14a}$$

$$1 + a' \coth\left(\frac{a'\Gamma'}{2}\right) = -|tq|\frac{\sin u}{\sin u'}\sqrt{1 - a'^2} \tag{14b}$$

for the primed region, and

$$\cos(u + u') = \frac{|t|^2 + |q|^2}{2|tq|}\sqrt{1 - a^2}, \tag{14c}$$

$$1 + a \coth\left(\frac{a\Gamma}{2}\right) = -\frac{|t|}{|q|}\frac{\sin u'}{\sin u}\sqrt{1 - a^2} \tag{14d}$$

for the unprimed region. $|q|^2 = |A_{40}|^2/|A'_{40}|^2$ is the ratio of input intensities and $|t|^2$ is the transmissivity of all loops involving external mirrors (equal to the product of mirror reflectivities). The device transmissivity and reflectivity are given by

$$T = \frac{|A'_{20}|^2}{|A_{40}|^2} = \frac{|A_{20}|^2}{|A'_{40}|^2} = |t|^2 \cos^2(u + u'), \tag{15a}$$

$$R = \frac{|A_{30}|^2}{|A_{40}|^2} = \frac{|t|^2}{|q|^2}\sin^2(u + u'). \tag{15b}$$

To determine the steady state of the device, one should solve equations (14) for the values of a, a', u and u'. However, to find the threshold and operation conditions one need not go through the solution.

The threshold is determined from the universal relation involving the parameters a', a, Γ' and Γ. Such a relation is obtained by combining equations (14*b*) and (14*d*):

$$\left[1 + a' \coth\left(\frac{a'\Gamma'}{2}\right)\right]\left[1 + a \coth\left(\frac{a\Gamma}{2}\right)\right] = |t|^2\sqrt{(1 - a'^2)(1 - a^2)}. \tag{16}$$

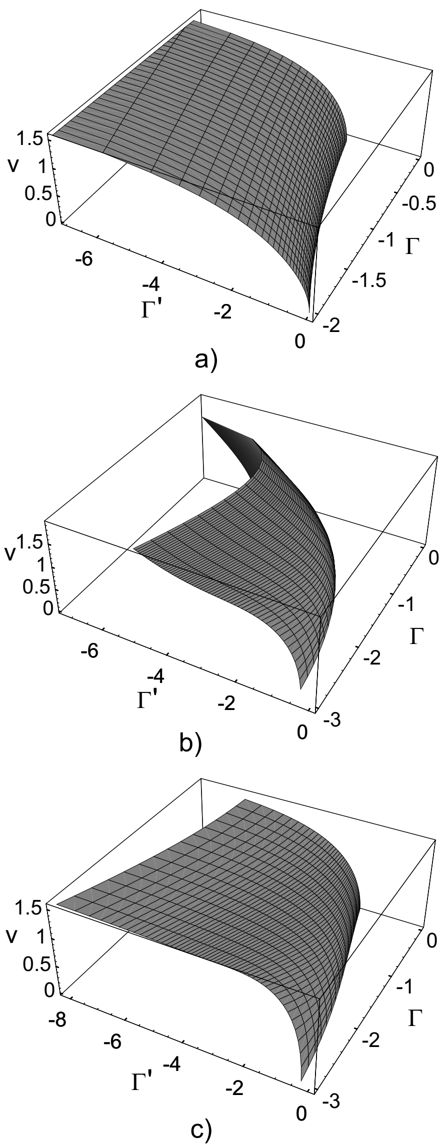

Figure 5. Total grating action $v = u + u'$ as a function of coupling strengths. (*a*) TG–TG ring, (*b*) RG–RG ring, (*c*) TG–RG ring. In all cases $|t| = 0.8$ and $|q| = 1$.

This relation provides the threshold curve in the plane of (Γ, Γ') parameters, once the minimum values of a' and a are found. They follow from equations (14*a*) and (14*c*) in the limit $u' \to 0$, $u \to 0$:

$$a'_{\rm th} = \frac{1 - |tq|^2}{1 + |tq|^2}, \qquad a_{\rm th} = \frac{|t|^2 - |q|^2}{|t|^2 + |q|^2}. \tag{17}$$

A sample of threshold curves is drawn in figure 4 for all three types of interconnected rings. A consequence of these results is that the threshold values of coupling strengths depend not only on the reflectivities of external mirrors, but also on the ratio of input intensities. Figure 5 presents the total grating action $v = u + u'$ as a function of Γ' and Γ. The threshold curves are found at the intersection of the action surface with the plane (Γ, Γ'). Figure 6 depicts the reflectivities along the line $\Gamma' = \Gamma$. We defer discussion of these results until the conclusions.

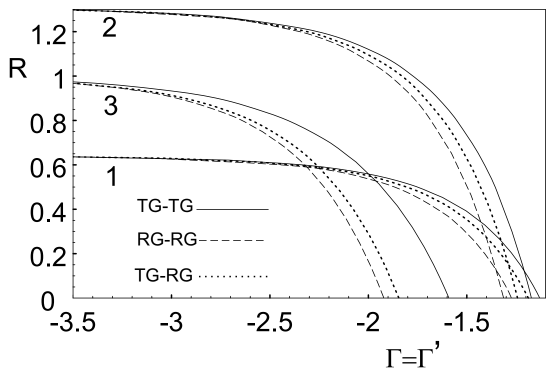

Figure 6. Reflectivity along the line $\Gamma = \Gamma'$ for the following parameters: (1) $|t| = 0.8$, $|q| = 1$; (2) $|t| = 0.8$, $|q| = 0.7$; (3) $|t| \approx 0.5$, $|q| = 0.5$.

Interesting special cases are for $t = 1$ or $q = 1$. One finds then that

$$\Gamma_{\rm th} + \Gamma'_{\rm th} = \frac{1+|q|^2}{1-|q|^2} \ln |q|^2, \tag{18a}$$

and

$$\exp(-a_{\rm th}\Gamma_{\rm th}) + \exp(-a_{\rm th}\Gamma'_{\rm th}) = \frac{1+|t|^2}{|t|^2}. \tag{18b}$$

In principle, one should not assign fixed values of Γ' and Γ to the crystal. They can vary due to material parameters and the geometry of mixing. There exist intervals of allowed values of Γ' and Γ, starting from the threshold and going to the maximum achievable under given conditions. The question is then at what values of Γ' and Γ the device will operate.

The answer is provided by Fermat's principle. The device should operate at the extremum value of $u' + u$. The analysis of equations (14) and (15) reveals that this should happen along the line $u' = u$, when the two IR contribute equally. The expressions for each IR then reduce to the case of the simple TG ring, with the provision of an additional parameter q. However, this does not mean $\Gamma' = \Gamma$. As long as $t \neq 1$ and $q \neq 1$, the device will operate at unequal values of the coupling strengths.

5. RG–RG ring

Application of equation (6) to both IRs leads to the relations

$$1 + \coth\left(\frac{\Gamma'}{2}\right) = \frac{2|tq|^2}{1+|tq|^2} \mathrm{sech}^2 u \left(\sinh^2 u - \frac{\sinh u}{\sinh u'}\right), \tag{19a}$$

$$1 + \coth\left(\frac{\Gamma}{2}\right) = \frac{2|t|^2}{|t|^2+|q|^2} \mathrm{sech}^2 u' \left(\sinh^2 u' - \frac{\sinh u'}{\sinh u}\right), \tag{19b}$$

and the transmissivity and the reflectivity are given by

$$T = \frac{|A_{1d}|^2}{|A'_{10}|^2} = \frac{|A'_{4d}|^2}{|A_{40}|^2} = |t|^2(\mathrm{sech}\, u' \mathrm{sech}\, u - \tanh u' \tanh u)^2, \tag{20a}$$

$$R = \frac{|A_{30}|^2}{|A_{40}|^2} = \frac{|t|^2}{|q|^2}(\tanh u' \mathrm{sech}\, u + \tanh u\, \mathrm{sech}\, u')^2. \tag{20b}$$

In this geometry $|q|^2 = |A_{40}|^2/|A'_{10}|^2$. There are no order parameters. The threshold curves follow from the relation

$$\left[1 + \coth\left(\frac{\Gamma'_{\rm th}}{2}\right)\right]\left[1 + \coth\left(\frac{\Gamma_{\rm th}}{2}\right)\right] = \frac{4|q|^2|t|^4}{(|t|^2+|q|^2)(1+|tq|^2)}, \tag{21a}$$

and a sample is presented in figure 4. The corresponding total grating action is represented in figure 5(b) and the reflectivity in figure 6. Equation (21a) can be transformed into an explicit dependence

$$\exp(\Gamma'_{\rm th}) = \frac{1 - \exp(\Gamma_{\rm th})}{1 + \exp(\Gamma_{\rm th})(|t|^2 + |q|^2 + |t|^2|q|^4)/|t|^2|q|^4}. \tag{21b}$$

Concerning the operation of the RG–RG ring, the most efficient working is again achieved at $u = u'$. One then recovers the simple RG ring formulae. Similar to the TG–TG ring, the steady-state operation need not proceed at $\Gamma' = \Gamma$.

6. TG–RG ring

Application of the grating action method to the mixed ring leads to mixed results. There are three equations, for u', a', and u:

$$\cos u' - \sin u' \sinh u = \frac{1+|tq|^2}{2|tq|} \cosh u \sqrt{1-a'^2}, \tag{22a}$$

$$1 + a' \coth\left(\frac{a'\Gamma'}{2}\right) = -|tq| \frac{\tanh u}{\sin u'} \sqrt{1-a'^2}, \tag{22b}$$

$$1 + \coth\left(\frac{\Gamma}{2}\right) = \frac{2|t|^2}{|t|^2+|q|^2} \frac{\sin u'}{\sinh u} (\sin u' \sinh u - \cos u'), \tag{22c}$$

while the expressions for the transmissivity and the reflectivity are of the form

$$T = \frac{|A_{4d'}|^2}{|A_{40}|^2} = |t|^2 \mathrm{sech}^2 u (\cos u' - \sin u' \sinh u)^2, \tag{23a}$$

$$R = \frac{|A_{30}|^2}{|A_{40}|^2} = \frac{|t|^2}{|q|^2} \mathrm{sech}^2 u (\sin u' + \cos u' \sinh u)^2. \tag{23b}$$

The universal relation

$$\left[1 + a' \coth\left(\frac{a'\Gamma'}{2}\right)\right]\left[1 + \coth\left(\frac{\Gamma}{2}\right)\right] = \frac{|t|^2(1+|tq|^2)}{|t|^2+|q|^2}(1-a'^2) \tag{24}$$

provides the dependence of a' on the coupling strengths Γ' and Γ. It also offers the threshold condition, once the minimum value of a', which is the same as in equation (17) for the TG–TG ring, is substituted. A few curves for different values of the parameters are presented in figure 4 and the total grating action is shown in figure 5(c). It is seen that the threshold curves approach those of the TG–TG ring at high values of Γ' and low values of Γ, and those of the RG–RG ring at low values of Γ' and high values of Γ. The reflectivity is shown in figure 6.

7. Conclusions

We have investigated theoretically the three types of interconected PR rings, introduced some time ago by Mamaev and Zozulya [6]. The TG–TG, RG–RG and mixed TG–RG geometries are considered, using the grating action method. The thresholds and reflectivities of all three geometries are found, and Fermat's principle invoked to determine the operation conditions. When one compares the thresholds and the reflectivities of the three interconnected rings, it is seen that the TG–TG ring is the most useful, the RG–RG is the least useful, and the TG–RG comes in between. The TG–TG ring possesses the lowest threshold and attains the highest reflectivity, as can be seen from figures 4 and 6. Exactly the opposite holds for the RG–RG ring.

In all the geometries, the threshold values of coupling strengths need not be fixed numbers, and may vary, depending on such parameters as the ratio of input intensities and the transmissivity of connection loops. The operation of these devices is governed by Fermat's principle. The TG–TG and RG–RG rings preferably work at the equal values $u' = u$ of grating actions as this leads to the maximum value of the reflectivity. It does not mean that the coupling constants at the operation point are equal. Depending on the values of mirror reflectivities and the ratio of input intensities, the devices start and operate at unequal coupling strengths in the two IRs.

Acknowledgment

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