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LETTER TO THE EDITOR

Dynamics of surfaces and a generalised nonlinear Schrödinger equation

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Abstract. Dynamics of surfaces is connected with various soliton equations, in particular with a recent generalisation of the nonlinear Schrödinger equation via an elementary account of the local geometry of moving frame fields and surfaces.

The recently introduced generalised nonlinear Schrödinger equation (GNLSE):

$$\mathrm{i}\dot{u}(x, t)+(fu)''+Ru=0, \qquad R=\int^{x}|u|(f|u|)', \tag{1}$$

has proven not only difficult to solve (Balakrishnan 1982a, b, Lakshmanan and Bullough 1980, Belić 1983), but also indispensable in the description of some peculiar physical processes (Balakrishnan 1982a, b, Belić 1983). In equation (1), $u(x, t)$ is assumed to be complex, $f=f(x)$ is real, the dot denotes the temporal, and the prime the spatial derivative. The solution of the GNLSE by the inverse scattering method requires an extension of the ZS-AKNS eigenvalue problem (Zakharov and Shabat 1972, Ablowitz *et al* 1974), in that then the eigenvalue was no longer simply a parameter in the theory, but a function of space and time with an evolution equation of its own (Balakrishnan 1982a, b).

In applications the GNLSE was found instrumental in the analysis of the Heisenberg spin chain with the site-dependent exchange integral (Balakrishnan 1982a, b) and in the description of the motion of inhomogeneous vortex filaments in a fluid (Belić 1983). In this letter, apart from noting that the GNLSE can be written in the form of the generalised Madelung fluid (Guerra 1981, Nonnenmacher *et al* 1983), we connect the GNLSE with the dynamics of a surface by use of a simple geometric argument. Actually, we present a general approach to soliton equations via dynamics of frame fields in a three-dimensional Euclidean space.

First, if $u=\sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\sigma}$ is substituted into the GNLSE, two equations follow

$$\dot{\sigma}+f(\sigma')^2=\rho^{-1/2}(f\sqrt{\rho})''+R, \qquad \dot{\rho}+2(\rho\sigma')'f=-4\rho\sigma'f'. \tag{2a, b}$$

In the language of fluid dynamics, (2a) represents the equation for the velocity potential, and (2b) the continuity equation of a generalised (one-dimensional) Madelung fluid. For f constant, we recognise expressions for the standard nonlinear Schrödinger-Madelung field (Nonnenmacher *et al* 1983).

Second, we consider dynamics of a frame field $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ by expressing the covariant derivative and the temporal derivative of these vector fields in terms of the vector fields

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themselves. For the spatial dependence of the frame we have (O'Neill 1971)

$$\begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \\ \hat{e}_3' \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}, \tag{3}$$

where here and thereafter the prime denotes the covariant derivative in a direction to be specified, and ω_{ij} are components of the connection forms of the frame. By defining

$$\Omega_i \equiv \varepsilon_{ijk}\omega_{jk}, \tag{4}$$

with ε_{ijk} denoting the Levi-Civita symbol, equation (3) can be rewritten in the form

$$\hat{e}_1' = \mathbf{\Omega} \times \hat{e}_1, \qquad \hat{e}_2' = \mathbf{\Omega} \times \hat{e}_2, \qquad \hat{e}_3' = \mathbf{\Omega} \times \hat{e}_3, \tag{5}$$

where $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$. Similarly, for the temporal dependence of the moving frame we can write:

$$\dot{\hat{e}}_1 = \boldsymbol{\omega} \times \hat{e}_1, \qquad \dot{\hat{e}}_2 = \boldsymbol{\omega} \times \hat{e}_2, \qquad \dot{\hat{e}}_3 = \boldsymbol{\omega} \times \hat{e}_3, \tag{6}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ stands for the angular velocity of the frame. The integrability conditions, $\hat{e}_{1xt} = \hat{e}_{1tx}$, etc, lead to the equation:

$$\dot{\mathbf{\Omega}} - \boldsymbol{\omega}' - \mathbf{\Omega} \times \boldsymbol{\omega} = 0. \tag{7}$$

This equation on the one hand contains many soliton equations, and on the other is connected with the ZS-AKNS two-component scattering problem (Lamb 1977, Lakshmanan 1979). In specifying the spatial derivative, we may connect the frame with various geometric objects. If, for example we choose the frame to be the Frenet trihedron of a curve, and specify the derivative to be taken along the curve, then Ω_1 represents the torsion of the curve, Ω_3 represents the curvature and Ω_2 equals zero. This approach has been advocated by Lamb (1977) and Lakshmanan (1979). Here we choose a surface as the geometric object, and denote by $\hat{e}_1$, $\hat{e}_2$, $\hat{u}$ the frame restricted to the surface, with $\hat{u}$ being the surface normal, and $\hat{e}_1$ and $\hat{e}_2$ unit vectors in the principal directions of the tangent plane. If the directional derivative is taken along $\hat{e}_1$, then Ω_1 and Ω_2 equal principal curvatures k_1 and k_2 of the surface, and Ω_3 represents the geodesic curvature γ. In this manner the frame is also restricted to a principal curve of the surface, and we may consider the dynamics of a string on the surface as well. Equation (7) in the component form now reads as follows:

$$\dot{k}_1 - \omega_1' - k_2\omega_3 + \gamma\omega_2 = 0, \tag{8a}$$

$$\dot{k}_2 - \omega_2' - \gamma\omega_1 + k_1\omega_3 = 0, \tag{8b}$$

$$\dot{\gamma} - \omega_3' - k_1\omega_2 + k_2\omega_1 = 0. \tag{8c}$$

This system of three equations contains six unknowns—it constitutes an incomplete set. So one can chose the ω_i in terms of k_1, k_2, γ in order to complete the set. This essentially means on the one hand restriction of the dynamics of the frame, and on the other restriction of the geometric characteristics of the surface. By an appropriate choice of ω_i various soliton equations can be obtained. For example, the integral soliton equation (Ablowitz *et al* 1974, Lamb 1977)

$$\mathrm{i}\dot{u} - u' - u \int^x |u|^2 = 0 \tag{9a}$$

is obtained by setting

$$\omega_1 = -\gamma'/\gamma, \qquad \omega_2 = -\gamma, \qquad \omega_3 = 0, \tag{9b}$$

and by assuming $u = \gamma \exp(\mathrm{i} \int^x k_1)$ and $k_2 = 0$. This choice is amenable to the AKNS two-component inverse scattering analysis.

For a description of GNLSE it turns out that we may select the principal curve to be the geodesic of the surface, i.e. the geodesic curvature may be set equal to zero. Furthermore, by making the following choice for ω_1, ω_2, ω_3:

$$\omega_1 = -k_1 k_2 f, \qquad \omega_2 = \frac{(k_1 f)''}{k_1} - k_2^2 f, \qquad \omega_3 = -(k_1 f)', \tag{10}$$

the first two equations of the system (8) become:

$$\dot{k}_1 + (f k_1 k_2)' + k_2 (k_1 f)' = 0, \tag{11a}$$

$$\dot{k}_2 - \left(\frac{(k_1 f)''}{k_1} - k_2^2 f\right)' - (k_1 f)' k_1 = 0, \tag{11b}$$

while the last equation becomes identity. However, if $u = k_1 \exp(\mathrm{i} \int^x k_2)$ is assumed, then (11) also represents the system of equations for the amplitude and the phase of the GNLSE (see equation (2) as well).

A connection with the AKNS procedure is established by considering any one component of the Darboux vector:

$$\varphi \equiv \frac{e_2 + \mathrm{i} e_3}{1 - e_1} = \frac{1 + e_1}{e_2 - \mathrm{i} e_3}, \tag{12}$$

where e_1, e_2, e_3 are some scalar components of the trihedral units. Using (5) and (6), two symmetric Riccati equations for the spatial and temporal dependence of φ are obtained:

$$\varphi' = -\mathrm{i}\Omega_1 \varphi + \tfrac{1}{2}(\Omega_3 - \mathrm{i}\Omega_2) + \varphi^2 \tfrac{1}{2}(\Omega_3 + \mathrm{i}\Omega_2) \tag{13a}$$

$$\dot{\varphi} = -\mathrm{i}\omega_1 \varphi + \tfrac{1}{2}(\omega_3 - \mathrm{i}\omega_2) + \varphi^2 \tfrac{1}{2}(\omega_3 + \mathrm{i}\omega_2). \tag{13b}$$

With $\varphi = v_2/v_1$ four linear equations of the scattering ZS problem follow (Lamb 1977, Lakshmanan 1979). From these we find the AKNS A, B, C, q, r coefficients:

$$A = \tfrac{1}{2}\mathrm{i}\left(\omega_1 - \int^x \dot{\Omega}_1\right), \qquad B = -\tfrac{1}{2}(\omega_3 + \mathrm{i}\omega_2) \exp\left(-\mathrm{i} \int^x (\Omega_1 - \zeta)\right) = -C^* \tag{14a}$$

$$q = -\tfrac{1}{2}(\Omega_3 + \mathrm{i}\Omega_2) \exp\left(-\mathrm{i} \int^x (\Omega_1 - \zeta)\right) = -r^*, \tag{14b}$$

where ζ is the scattering (eigenvalue) parameter.

In closing we note that this procedure for connecting the system (8) with the two-component ZS eigenvalue problem will not work for the GNLSE (though it works for other soliton equations, the NLSE for instance). An extension of the AKNS procedure was found necessary. This problem, however, has been considered elsewhere (Balakrishnan 1982a, b, Lakshmanan and Bullough 1980, Belić 1983), and will not be addressed here.

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