Two- and Four-Wave Mixing with Saturable Absorption and Gain

Dejan Timotijević, Milivoj Belić, and Robert W. Boyd, *Member, IEEE*

Abstract—**An exact solution to a model of two- and four-wave mixing in photorefractive media with saturable gain and absorption is presented. Pump depletion effects are accounted for, and the procedure for matching two-point boundary conditions is given. Possibilities of multistable solutions are investigated, and procedures on how to deal with such situations are outlined. It is found that the energy transfer between waves is less effective in the nonsaturated regime as compared to the saturated regime. It is also established that the nonsaturated system is more stable than the saturated under the same conditions.**

I. Introduction

A number of articles over the last decade were concerned with the theory and exact solution of wave equations describing photorefractive (PR) optical phase conjugation (OPC) [1]. Thus far they have been solved only in the saturation regime [2]. This regime is established under high illumination intensities, when most of the trapping sites for photoinduced charges are occupied, and the wave coupling constant g and absorption coefficient α do not depend on the light intensity. However, nowadays the experiments with photorefractives are done at low power levels, and with the pump depletion and absorption taken into account, the conditions for saturation are not met. Indeed, it is observed that at low beam intensities the wave coupling and the absorption coefficient in barium titanate become intensity dependent [3], [4]. It is now known in what way this dependence affects the phase conjugating (PC) process.

The models with intensity-dependent coupling or intensity-dependent absorption have already been the subject of scientific investigation [4]–[8]. However, to date there is no complete, fully consistent theory; this is no wonder, in view of the complexity of the PR effect. In the situation where experimental results vary from sample to sample, and there are many free parameters left to fit the theory, we utilize simple models which capture the essential features of intensity dependence: initial linear growth and saturation.

An early such model was given by Townsend and LaMacchia [4]. According to this model the intensity dependent two-wave mixing (2WM) gain coefficient is given by:

$$g(I) = \frac{g_s I}{a + b} \tag{1}$$

where g_s is the saturated gain, $a = (c\aleph IN + b^2)^{1/2}$ and $b = \aleph I/2 + 1/2\tau_0$. Here c is the capture coefficient for the photoexcited charges, $\aleph I$ is their generation rate, N is the total number density of charges available for excitation, and τ_0 is the thermal decay constant. The model predicts a linear dependence of $g(I)$ for low intensities, and a saturation for high.

The theory currently in vogue comes from Kukhtarev *et al.* [5]. According to this theory, as well as the similar hopping theory of Feinberg *et al.*, [6] the expression for the saturated space-charge field, or the saturated gain which is proportional to the space-charge field, should be multiplied by an intensity dependent factor $R(I)$, given by:

$$R(I) = \frac{\sigma_h - \sigma_e}{\sigma_h + \sigma_e + \sigma_d} \tag{2}$$

where σ_h, σ_e, σ_d are the hole, the electron, and the dark (thermal) conductivities of the crystal, respectively. Both the hole and the electron photoconductivities vary linearly with the illumination intensity. Even though experimental results [7] on $BaTiO_3$ crystal suggest sublinear dependence, many features of the PR effect are adequately described by the Kukhtarev theory.

A recent extension of the Kukhtarev theory, which takes into account shallow traps in addition to deep donors, [7], [8] similarly predicts that the Kukhtarev expression for the saturated space-charge field, and therefore the 2WM gain coefficient, should be multiplied by an intensity dependent factor:

$$\eta(I) = \frac{1}{k_O^2}\left(k_{OD}^2 + \frac{k_{OT}^2}{1 + \beta/s_T I}\right) \tag{3}$$

where $k_O^2 = k_{OD}^2 + k_{OT}^2$ is the square of the total Debye screening wave vector, and k_{OD}^2 and k_{OT}^2 are the portions coming from deep donors and shallow traps. β is the thermal excitation rate and s_T is the light excitation cross section for traps. The Debye screening wave vector also depends on the intensity. However, when the donor contribution to the Debye screening is predominant, this dependence is supressed, $\eta \to 1$. On the other hand, when the trap screening is predominant, or when the two con-

Manuscript received April 10, 1991; revised March 2, 1992.

D. Timotijević and M. Belić are with the Institute of Physics, 11001 Belgrade, Yugoslavia.

R. W. Boyd is with The Institute of Optics, University of Rochester, Rochester, NY 14627.

IEEE Log Number 9201382.

tribute about equally, a simple functional dependence is obtained:

$$\eta(I) = \frac{I}{I + \beta/s_T} \quad \text{or} \quad \eta(I) = \frac{1}{2}\left(1 + \frac{I}{I + \beta/s_T}\right). \tag{4}$$

This theory is applicable to both n-type PR crystals, such as BSO, or to p-type crystals, such as $BaTiO_3$. It also agrees well with experimental results [7].

Thus, by inspecting (1)–(4), a simple generic intensity dependence of the gain coefficient could be suggested:

$$g(I) = \frac{g_s I}{I + C} \tag{5}$$

including simple variations of the same. Here C is a parameter dependent on the material. We use (5) as the basic model. It can be tested experimentally, for example by external incoherent illumination of the crystal. The saturable gain then has exactly the form given by (5), and the constant C corresponds to the intensity of this incoherent illumination.

The wave mixing models employed here are the standard two- and four-wave mixing arrangements in PR crystals [1]. In 2WM a reflection geometry is presumed, with the two waves (A_1 and A_2) incident on the crystal from the opposite sides. In 4WM there are additional two waves, the probe A_4 impinging from the side of the wave A_1, and its counterpropagating PC replica A_3. Our goal is to consider PR mixing of these waves with saturable coupling constants, and at the same time to include intensity dependent absorption, which is appropriate to PR crystals. We also investigate the possibilities of unstable behavior in the system.

There are four ways in 4WM in which the waves can mix and build diffraction gratings through the PR effect. Two of these contribute to the PC wave generation, and are commonly denoted as the transmission and the reflection gratings. The other two come from the 2WM of the pumps, and of the probe with the PC wave. Most accounts up to date assume only one important grating (transmission or reflection) to be operative, and ignore 2WM terms. We assume that both transmission and reflection gratings are present, and contribute equally, and we account for the mixing of the pumps as the predominant 2WM process.

The layout of the paper is as follows. Section II deals with 2WM, Section III deals with 4WM, and Section IV summarizes our results.

II. Two-Wave Mixing

A thorough recent review of 2WM in nonlinear media is provided by Yeh [9]. Reflection geometry with and without intensity-dependent coupling was first considered by Ja [10]. Belić [11] included linear absorption exactly. A similar procedure will be used here to treat the intensity-dependent coupling and nonlinear absorption.

Our starting points are the wave equations for intensities in the two-wave PR mixing [9]:

$$I_1' + \alpha(I) I_1 + 2g(I)\frac{I_1 I_2}{I} = 0 \tag{6a}$$

$$I_2' - \alpha(I) I_2 + 2g(I)\frac{I_1 I_2}{I} = 0 \tag{6b}$$

where $I = I_1 + I_2$ is the total intensity, and $\alpha(I)$ and $g(I)$ are the intensity-dependent absorption and wave coupling coefficients. The prime denotes the derivative along the propagation (z) direction. For $g(I)$ we pick the expression given by (5) (dropping the subscript s). Many different models exist for the saturable absorption coefficient. We opt for a form found by Brost, Motes, and Rotge [3] to account for the observed behavior in PR crystals:

$$\alpha(I) = \alpha_0 + \alpha\frac{I}{C + I} \tag{7}$$

where α_0 is the linear absorption, and α is a material parameter. In principle, the constant C here could be different from the one in $g(I)$. However, there is no reason to presume that the saturation mechanism in PR gain is fundamentally different from the one in PR loss. We assume that the constant C is the same in both (5) and (7). In order to consider only the effects of saturable absorption, linear absorption is neglected.

The solution of (6) proceeds as follows [11]. First, a set of new variables is introduced: $u = I_1 + I_2$, $v = I_1 - I_2$, $f = 2(I_1 I_2)^{1/2}$. In terms of these variables, equations to be solved become:

$$(u + C)u' + \alpha u v + g f^2 = 0 \tag{8a}$$

$$(u + C)v' + \alpha u^2 = 0 \tag{8b}$$

$$(u + C)f' + \alpha u f = 0. \tag{8c}$$

Since $f^2 = u^2 - v^2$, only two of these equations are independent. One equation is eliminated by assuming:

$$u = f\cosh F \qquad v = f\sinh F. \tag{9}$$

Equations for the new independent variables f and F are:

$$(C + f\cosh F)F' = f(g\sinh F - \alpha\cosh F) \tag{10a}$$

$$(C + f\cosh F)f' = -g f^2\cosh F. \tag{10b}$$

In this form the problem can be treated analytically. Dividing (10a) by (10b) brings an implicit relation between F and f:

$$R_1(F, F_O) = \int_{F_O}^{F}\frac{dx}{\beta - \tanh x} = \ln\frac{f}{f_O} \tag{11}$$

where $\beta = \alpha/g$ and the subscript O means that the corresponding quantity is evaluated at $z = 0$. It is presumed that the crystal extends from $z = 0$ to $z = d$, and that $I_1(z = 0) = C_1$ and $I_2(z = d) = C_2$ are the given boundary conditions. A convenient logarithmic expression exists for the integral in (11), $R_1 = \ln[G(F)/G(F_O)]$, with:

$$G(F) = |\Gamma\exp 2F + 1|^{1/(\beta^2 - 1)}\exp\left(\frac{F}{\beta + 1}\right) \tag{12}$$

where $\Gamma = (\beta - 1)/(\beta + 1)$. The values $\beta = \epsilon = \pm 1$, i.e., $g = \pm\alpha$, require special analysis, and the result is $G(F) = \exp[\epsilon F/2 + \exp(2\epsilon F)/4]$. There is no discontinuity as $\beta \to \pm 1$.

The knowledge of the relation between F and f allows an easy integration of (10a),

$$R_2(F, F_O) - R_1(F, F_O) = gz \tag{13}$$

with R_2 denoting another quadrature involving variable $f(F)$:

$$R_2(F, F_O) = C \int_{F_O}^{F} \frac{dx}{f(x)\,(\sinh x - \cosh x)}$$

$$= -D \int_{F_O}^{F} \frac{\exp[\beta x/(\beta + 1)]}{|h(x) + 1|^q \operatorname{sgn}(h + 1)}\, dx \tag{14}$$

where $D = [2CG(F_O)/(\beta + 1)f_O]$, $h(F) = \Gamma \exp(2F)$, $q = \beta^2/(\beta^2 - 1)$, and sgn denotes the sign function. In this manner, (13) defines the function $F(z)$, or explicitly $z(F)$, and (11) defines the function $f(F) = f_O G(F)/G(F_O)$. The intensities are then given by $I_1 = f \exp F/2$ and $I_2 = f \exp(-F)/2$, and the equation part of the problem is finished. There remains the boundary-value part of the problem.

Relations (11) and (13) are simply connected with the incomplete beta function, or Gausses hypergeometric function [12]. For $|\beta| > 1$ ($\Gamma > 0$), the integral in (14) can be expressed as:

$$R_2 = -D \int_{h_O}^{h} \frac{x^{\lambda - 1}}{(x + 1)^q}\, dx$$

$$= D'[F(h, \beta) - F(h_O, \beta)] \tag{15}$$

where $\lambda = \beta/[2(\beta + 1)]$, $D' = D|\Gamma|^{-\lambda}$, and

$$F(h, \beta) = \frac{h^{-\mu}}{\mu}\, {}_2F_1\left(\frac{2 - \beta}{2(1 - \beta)}, \mu; \mu + 1; -\frac{1}{h}\right). \tag{16}$$

Here ${}_2F_1$ denotes the Gauss hypergeometric function, and $\mu = 1/(1 - \beta^2)$. Equivalently, by a variable change $r = h/(h + 1)$, the same integral can be expressed as:

$$R_2 = -D' \int_{r_O}^{r} x^{\lambda - 1}(1 - x)^{q - \lambda - 1}\, dx$$

$$= D'[B_r(\lambda, q - \lambda) - B_{r_O}(\lambda, q - \lambda)] \tag{17}$$

where $B_r(x, y)$ is the incomplete beta function [12]. For $|\beta| < 1$ an expression similar to (15) holds:

$$R_2 = D'\left[F\begin{pmatrix} x, \beta \\ y, -\beta \end{pmatrix} - F\begin{pmatrix} x_O, \beta \\ y_O, -\beta \end{pmatrix}\right] \tag{18}$$

with $x = t/(t - 1)$, $y = 1/(t - 1)$, and $t = -h$. The upper pair of variables applies when $t < 1$, and the lower when $t > 1$.

The boundary-value part of the problem is handled in the following manner. In order to write down the solutions for intensities explicitly, the knowledge of F_O and F_d (i.e., t_O and t_d) in terms of the given boundary values

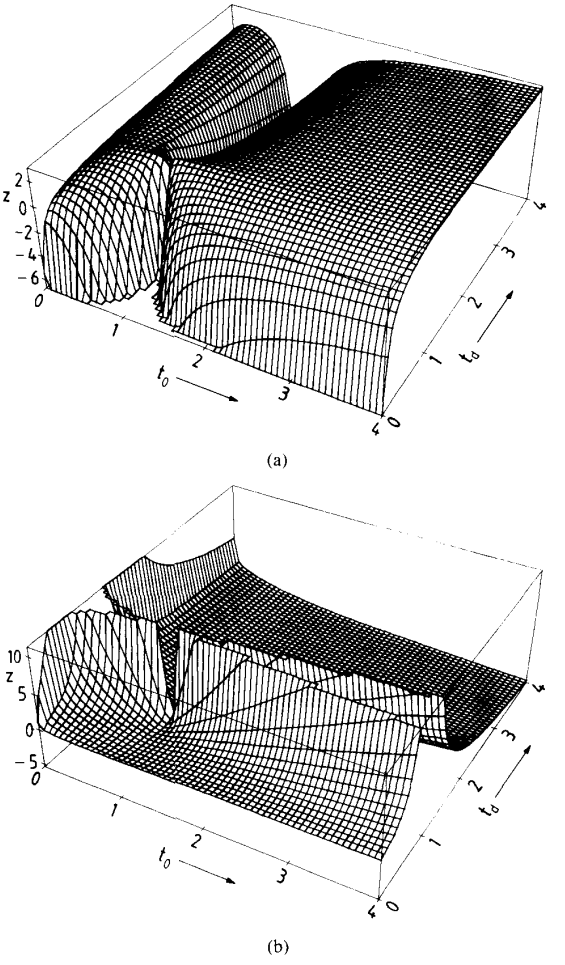


Fig. 1. (a) The surface $z = R_1(t_O, t_d) - R_2(t_O, t_d) + gd$, (19a). Here and thereafter $d = 1$ cm. (b) The corresponding surface (19b). The cross sections of these two surfaces with the $z = 0$ plane define multiple solutions.

C_1 and C_2 is necessary. They are found by solving a system of two coupled algebraic equations:

$$R_2(F_d, F_O) - R_1(F_d, F_O) - gd = 0 \tag{19a}$$

$$\ln\frac{G(F_O)}{G(F_d)} + F_O + F_d + \ln\frac{C_2}{C_1} = 0. \tag{19b}$$

As it happens often with nonlinear equations and implicit relations, (19) allows multiple solutions. Typical situation is presented in Figs. 1 and 2. Such solutions might complicate the analysis, especially when there is no clear criterion which would distinguish between the physically relevant (allowed) and irrelevant solutions [11]. Here, by simple inspection, a class of solutions is discarded. However, in the following section on 4WM, there is no such criterion, and multiple solutions appear, leading to a chaotic response of the system.

Fig. 3 presents a comparison between the saturated and the nonsaturated regime. It is seen that the saturated regime enables more efficient energy transfer from the wave

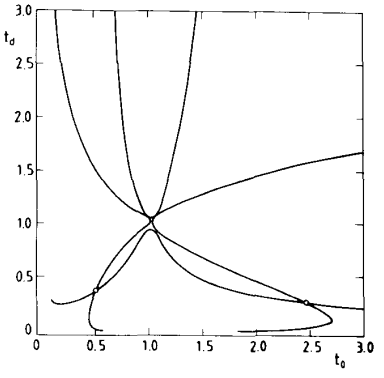


Fig. 2. Possible multiple solutions of (19). The curves presented are the cross cuts of surfaces in Fig. 1, and the points where these curves cut each other lead to multiple solutions. These points define the values of the arguments t_O and t_d for which the boundary conditions are satisfied. Of the two points visible, the point to the left leads to negative z, and should be discarded. The unique solution corresponds to the point to the right. Note also the "avoided crossing" nature of the singularity at $t_O = t_d = 1$.

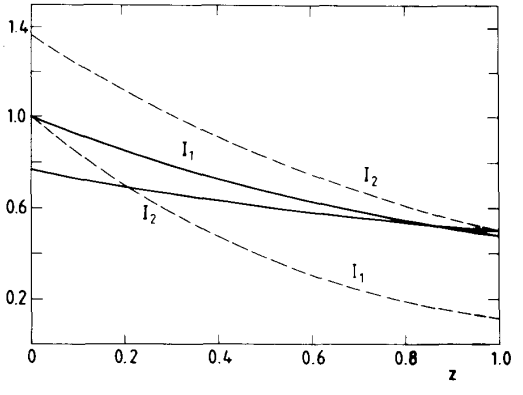


Fig. 3. Unique solutions to 2WM with nonsaturated (full lines) and saturated (dashed lines) coupling and absorption. The parameters are: $g = 1$ cm^{-1}, $\alpha = 0.5$ cm^{-1}, $C_1 = 1$, and $C_2 = 0.5$.

I_1 to the wave I_2 (for this sign of g; the opposite sign changes the direction of the energy flow.) A summary of our findings is provided in Section IV.

III. Four-Wave Mixing

The 4WM model employed here presumes that the formation of gratings proceeds by more than one grating mechanism. Usually the transmission gratings are preferred experimentally, and their predominance is assured by the choice of the direction of the c axis, or by other means. When no attention is paid to these matters, it must be assumed that competitive gratings are formed in the crystal, and their cumulative influence on the PC process must be taken into consideration.

The intensity equations describing 4WM in PR media, with the assumption that transmission and reflection gratings contribute equally (i.e., have the same g coefficient) are derived in [13]:

$$I_1' = 2\gamma(I)I_1I_2 - 2g(I)I_1(I_4 - I_3) \tag{20a}$$

$$I_2' = 2\gamma(I)I_1I_2 + 2g(I)I_2(I_4 - I_3) \tag{20b}$$

$$I_3' = 2g(I)I_3(I_1 + I_2) + 4g(I)(I_1I_2I_3I_4)^{1/2} \tag{20c}$$

$$I_4' = 2g(I)I_4(I_1 + I_2) + 4g(I)(I_1I_2I_3I_4)^{1/2} \tag{20d}$$

where the γ term describes 2WM between the pumps, and the g term takes into account the equal-strength 4WM. I is the total intensity, $I = I_1 + I_2 + I_3 + I_4$, and both $g(I)$ and $\gamma(I)$ have the form given by (5). In writing (20) it is assumed that diffusion is the dominating charge transporting process, and absorption is neglected. It seems impossible to solve coupled 4WM equations analytically with the absorption included in any form. Further, the relative phase $\phi = \phi_1 + \phi_2 - \phi_3 - \phi_4$ is locked to 0 or π (so-called exact OPC), and the gratings between the signal I_4 and the PC wave I_3 are ignored.

The solution of (20) proceeds similarly to the 2WM case. New variables involving partial sums and differences of the intensities are introduced: $u_1 = I_1 + I_2$, $u_2 = I_3 + I_4$, $v_1 = I_2 - I_1$, $v_2 = I_4 - I_3$, and a set of two auxiliary functions is defined: $f_1^2 = 4I_1I_2$, $f_2^2 = 4I_3I_4$. Equations to be solved become:

$$(C + I)u_1' = 2gv_1v_2 + \gamma f_1^2 \tag{21a}$$

$$(C + I)v_1' = 2gu_1v_2 \tag{21b}$$

$$(C + I)u_2' = 2g(u_1u_2 + f_1f_2) \tag{21c}$$

$$(C + I)v_2' = 2gu_1v_2. \tag{21d}$$

It is seen that the variables v_1 and v_2 are simply related: $v_1 = v_2 + \Delta$, where Δ is a constant to be evaluated from the boundary conditions. Similar equations can be written for f_1 and f_2:

$$(C + I)f_1' = \gamma f_1u_1 \tag{22a}$$

$$(C + I)f_2' = 2g(u_1f_2 + u_2f_1) \tag{22b}$$

and f_1 can also be expressed in terms of v_2: $f_1 = f_{1d}(v_2/v_{2d})^{\gamma/2g}$. Thus, $u_1 = (v_1^2 + f_1^2)^{1/2}$ is also given in terms of v_2. The strategy is to express the remaining variables u_2 and f_2 also in terms of v_2, and then to solve an equation for v_2. This is accomplished most easily by introducing a new variable w:

$$u_2 = v_2 \cosh w \qquad f_2 = v_2 \sinh w \tag{23}$$

and by rescaling all variables with respect to $v_{2d} = I_{4d}$: $u = u_1/v_{2d}, f = f_1/v_{2d}, v = v_2/v_{2d}$. These variables should not be confused with the ones used in the previous section. The final set of equations is of the form:

$$(c + i)w' = 2gf \qquad (c + i)v' = 2guv \tag{24}$$

where $c = C/v_{2d}$, $i = u + v \cosh w$ is the total intensity in new variables, $f(v) = av^{\xi/2}$, and $u(v) = [(v + \delta)^2 + f^2]^{1/2}$. Here $a = f_{1d}/v_{2d}$, $\xi = \gamma/g$, and $\delta = \Delta/v_{2d}$.

The set of coupled equations (24) can be solved in quadratures. The solution is of the form:

$$w(v) = \int_1^v \frac{f(x)}{xu(x)}\,dx \tag{25a}$$

$$\ln v + c \int_1^v \frac{dx}{xu(x)} + \int_1^v \frac{\cosh w(x)}{xu(x)}\, dx = 2g(z - d) \tag{25b}$$

and the convenient feature of the method is that the original two-point boundary value problem is transformed into an initial value problem. The values of both variables are known on the $z = d$ face of the crystal: $v_d = 1$, $w_d = 0$. However, the input parameters a and δ, which figure in (25), depend on the missing boundary values, and their evaluation is provided by a self-consistent procedure. The procedure is described in our recent publication [14].

In short, a and δ could be given either in terms of the missing boundary values I_{1d} and I_{4d} on the $z = d$ face of the crystal, or in terms of I_{20} and I_{30}, missing on the $z = 0$ face. The problem of fitting boundary conditions is resolved by an iteration map. One starts by choosing arbitrary initial values a_O and δ_O; from these I_{1d} and I_{4d} are calculated, and I_{20} and I_{30} (i.e., v_O and w_O) are found by evaluating integrals in (25), or by solving (24) on computer. This enables the calculation of the new values for a_1 and δ_1, and the procedure is repeated until convergence.

We find the procedure to be stable for g negative, and for arbitrary other parameters. For g positive the instabilities set in, and we investigate these instabilities by standard methods of nonlinear dynamics. This is done by evaluating the fixed points of the arbitrary composition power of the map. Such fixed points correspond to different periodic cycles that may exist in the map, and may reveal the nature of the transition to chaos, if there is one in the system. However, seeing such instabilities in a model map does not mean that they exist in a real crystal. We have addressed these questions elsewhere [14].

Figs. 4 and 5 show stable solutions for $C = 0$ and $C = 1$, and for two different values of γ. It is seen that the saturable regime is less effective in transferring energy between the waves than the saturated regime. The PC reflectivity attained is higher in the saturated regime. It is also seen that the parameter γ has little influence on the PC process. However, it strongly affects the pumps (as it should, since it is responsible for the 2WM between the pumps). The form of the functions v and w is also little affected by γ. The intensities are given in terms of v and w by the following formulas:

$$I_1 = I_{4d} \frac{u(v) - (v + \delta)}{2} \qquad I_2 = I_{4d} \frac{u(v) + (v + \delta)}{2} \tag{26a}$$

$$I_3 = I_{4d} v \frac{\cosh w - 1}{2} \qquad I_4 = I_{4d} v \frac{\cosh w + 1}{2}. \tag{26b}$$

Intensity reflectivity is thus:

$$R = \frac{\cosh w_O - 1}{\cosh w_O + 1} \tag{27}$$

and it is smaller than 1. This restriction arises because the case considered is of two competitive gratings set up simultaneously in the crystal [13]. Such competitive gratings may adversely affect other PC processes in PR crystals as well. For example, double phase conjugation is impossible under these conditions [14].

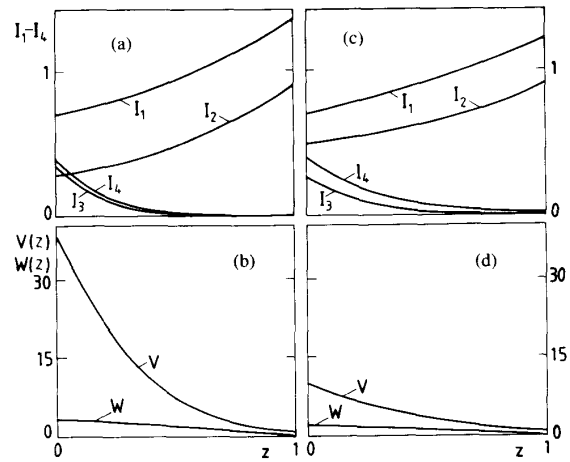


Fig. 4. Stable solutions for 4WM with saturated, (a) and (b), and nonsaturated, (c) and (d), coupling coefficients. (a) and (c) depict intensities, and (b) and (d) show the corresponding v and w functions. The nonsaturated case leads to a less effective phase conjugation. The reflectivities measured are: $R \cong 0.881$ for the saturated and $R \cong 0.657$ for the nonsaturated case. The parameters are: $g = -2\ \text{cm}^{-1}$, $\gamma = 1\ \text{cm}^{-1}$, $C_1 = 0.7$, $C_2 = 0.9$, $C_4 = 0.4$

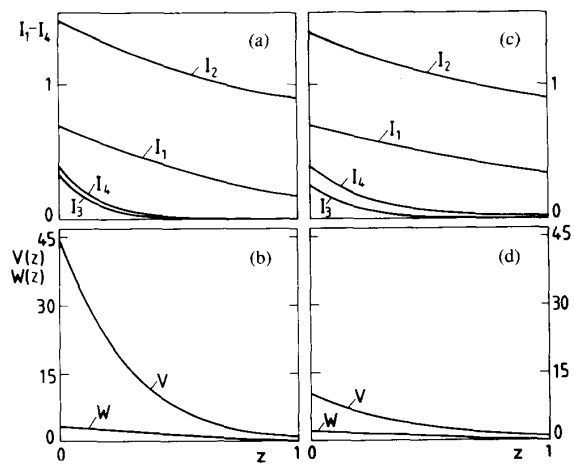


Fig. 5. The same as Fig. 4, only $\gamma = -1\ \text{cm}^{-1}$ in this case. The sense of the energy transfer for the pumps is reversed now, but there is little change in the PC process. The reflectivities are now $R \cong 0.856$ for the saturated and $R \cong 0.628$ for the nonsaturated case.

Figs. 6 and 7 display unstable situations that may arise in the system. Fig. 6(a) presents a bifurcation diagram of the reflectivity for the saturated crystal, and Fig. 6(b) represents the corresponding situation in the nonsaturated crystal. While in Fig. 6(a) complicated behavior is observed, including a transition to chaos via the (reverse) period doublings, in Fig. 6(b) only simple periodic orbits are visible. It is seen that the process of 4WM is more stable in the nonsaturated regime, i.e., stronger couplings

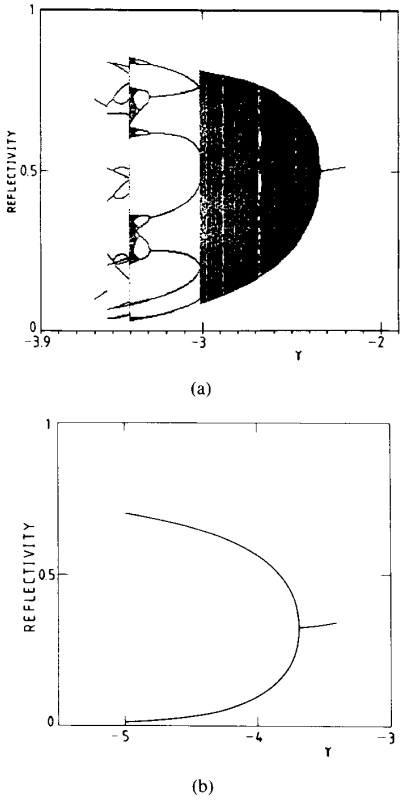


Fig. 6. (a) Bifurcation diagram of the intensity reflectivity as γ is varied, for the saturated regime ($C = 0$). To the right a quasi-periodic behavior of the system is observed. To the left the quasi-periodic behavior is interrupted by a chaotic window. Going to the left, the chaos is reached by a period doubling scenario. (b) The corresponding situation in the saturable regime ($C = 1$). No chaos is observed, only simple $P2$ and $P1$ periodic orbits.

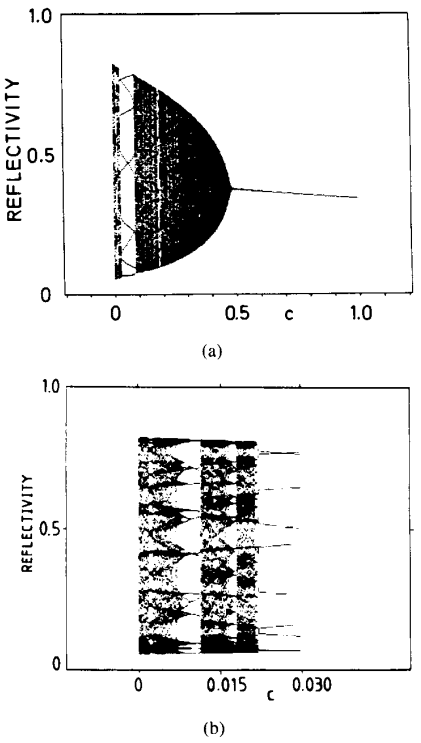


Fig. 7. Bifurcation diagram as the saturation parameter C is varied. The system is chaotic for $C = 0$ (saturation), and then proceeds through a series of changes until a unique solution is obtained for $C = 1$ (nonsaturated system). (b) depicts in greater detail the region around $C = 0$ from (a).

are needed to destabilize a nonsaturated system. This is further corroborated by Fig. 7, which depicts the bifurcation diagram obtained by sampling the constant C as the control parameter. Evidently, the system becomes more stable as the constant C is increased, which is equivalent to having less and less saturated crystal.

IV. Conclusion

We have investigated 2WM and 4WM processes in PR crystals in the nonsaturated regime, i.e., when the coupling "constant" and absorption depend on the light intensity. A model of saturable absorption in 2WM via reflection grating is introduced, and the corresponding wave equations are solved exactly. The solutions are written in terms of hypergeometric functions, and presented graphically. From the solutions it is evident that the energy transfer between waves is less effective in the nonsaturated stage of the process. No instabilities in 2WM are found.

In the second part of the paper a model of multigrating 4WM is investigated, with intensity-dependent coupling coefficients. A system of four nonlinear differential equations for the steady-state energy transfer is solved in terms of quadratures, and a boundary-value fitting procedure devised in the parameter space. Stable and unstable solutions are found, depending on the strength of the coupling.

For g negative, only stable solutions exist. In this region our procedure rapidly and accurately converges to a unique solution satisfying given boundary conditions. As compared to standard shooting or other methods, our procedure is found superior, especially when multiple solutions occur in the system.

We find that the energy transfer is adversely affected when the mixing of waves proceeds in the nonsaturated regime. This conclusion is perhaps obvious, however, we also find that the stability of the PC process is enhanced in this regime. The stability is also enhanced by turning the absorption on, but this influence is expected. Globally, absorption suppresses instabilities, but it also suppresses the processes of interest.

For g positive, sooner or later, instabilities set in. Different types of unstable behavior are observed: quasi-periodic motion on a torus, and a period doubling cascade to chaos. Chaotic behavior in this context means that the intensity reflectivity does not settle onto any particular value, but wanders on a strange attractor in the parameter space. In general, the nonsaturated regime is more stable than the saturated regime. In this respect Fig. 7 is partic-

ularly instructive: while for $C = 0$ (saturation) the system is chaotic, for $C = 1$ it is stable for the same set of parameters and boundary values.

In the end, it should be pointed out that instabilities found in a steady-state numerical analysis (such as ours), should be viewed with suspicion. They may not exist in the real crystal. The existence of such instabilities should be verified experimentally.

References

[1] P. Gunter and J. P. Huignard, Eds., *Photorefractive Materials and their Applications I and II*, Vols. 61 and 62 of Topics in Applied Physics. Berlin: Springer, 1988.

[2] M. Cronin-Golomb, B. Fisher, J. O. White, and A. Yariv, "Theory and applications of four-wave mixing in photorefractive media," *IEEE J. Quantum Electron.*, vol. QE-20, pp. 12-30, 1984; M. R. Belić, "Exact solution to the degenerate four-wave mixing in reflection geometry in photorefractive media," *Phys. Rev.*, vol. A31, pp. 3169-3174, 1985.

[3] G. A. Broost, K. A. Motes, and J. R. Rotge, "Intensity-dependent absorption and photorefractive effects in barium titanate," *J. Opt. Soc. Amer. B*, vol. 5, pp. 1879-1885, 1988.

[4] R. L. Townsend and J. T. LaMacchia, "Optically induced refractive index changes in $BaTiO_3$," *J. Appl. Phys.*, vol. 41, pp. 5188-5192, 1970.

[5] N. V. Kukhtarev, V. B. Markov, S. G. Odulov, M. S. Soskin, and V. L. Vinetskii, "Holographic storage in electrooptic crystals. I. Steady state," *Ferroelect.*, vol. 22, pp. 949-960, 1979.

[6] J. Feinberg, "Altering the photorefractive properties of $BaTiO_3$ by reduction and oxidation at 650°C," *J. Opt. Soc. Amer. B*, vol. 3, pp. 283-291, 1986.

[7] S. Ducharme and J. Feinberg, "Speed of the photorefractive effect in a $BaTiO_3$ single crystal," *J. Appl. Phys.*, vol. 56, pp. 839-842, 1984; D. Mahgerefteh and J. Feinberg, "Explanation of the apparent sublinear photoconductivity of photorefractive barium titanate," *Phys. Rev. Lett.*, vol. 64, p. 2195, 1990.

[8] P. Tayebati and D. Mahgerefteh, "Theory of the photorefractive effect for $Bi_{12}SiO_{20}$ and $BaTiO_3$ with shallow traps," *J. Opt. Soc. Amer. B*, vol. 8, pp. 1053-1064, 1991; *Erratum*, vol. B9, p. 177, 1992.

[9] P. Yeh, "Two-wave mixing in nonlinear media," *IEEE J. Quantum Electron.*, vol. 25, pp. 484-519, 1989.

[10] Y. H. Ja, "Energy transfer between two beams in writing a reflection volume hologram in a dynamic medium," *Opt. Quantum Electron.*, vol. 14, pp. 547-556, 1982; ——, "Intensity dependence of stationary energy transfer in degenerate two-wave mixing in a reflection geometry with photorefractive crystals," *Opt. Quantum Electron.*, vol. 17, pp. 291-295, 1985.

[11] M. R. Belić, "Comment on using the shooting method to solve boundary-value problems involving coupled-wave equations," *Opt. Quantum Electron.*, vol. 16, pp. 551-557, 1985; W. Krolikowski and M. R. Belić, "Multigrating phase conjugation: exact results," *Opt. Lett.*, vol. 13, pp. 149-151, 1988.

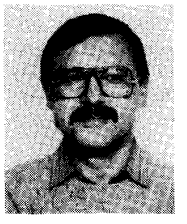
[12] A. Erdelyi, Ed., *Higher Transcendental Functions*. New York: McGraw-Hill, 1953, vol. 1.

[13] M. R. Belić, "Phase conjugation via multiple gratings in photorefractive crystals," *Phys. Rev. A*, vol. 37, pp. 1809-1812, 1988; M. R. Belić and W. Krolikowski, "Multigrating optical phase conjugation: numerical results," *J. Opt. Soc. Amer. B*, vol. 6, pp. 901-909, 1989.

[14] M. R. Belić, D. Timotijević, and W. Krolikowski, "Multigrating phase conjugation: chaotic results," *J. Opt. Soc. Amer. B*, vol. 8, pp. 1723-1731, 1991.

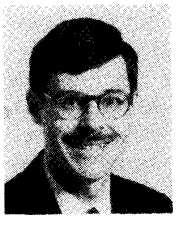
Dejan Timotijević was born in Yugoslavia in 1964. He received the B.S. degree in physics in 1987 and the M.S. degree also in physics in 1991 from the University of Belgrade, Yugoslavia.

He is currently working towards the Ph.D. degree in physics, on the problems of optical phase conjugation and photorefractive effect. His areas of interest include nonlinear dynamics and nonlinear optics.



Milivoj Belić received the B.S. degree in physics from the University of Belgrade, Yugoslavia, in 1974 and the Ph.D. degree also in physics from City College of New York, NY, in 1980,

He spent 1980 and 1981 as a research associate with the Optical Sciences Center, University of Arizona, Tucson, and 1986 and 1987 as a Humboldt Fellow at the Max-Planck Institute for Quantum Optics, Garching, Germany. He is currently an associate professor at the Institute of Physics, Belgrade, Yugoslavia. His research interests include phase conjugation, instabilities and chaos in optical system, and solitons—in general, nonlinear optics and nonlinear dynamics.



Robert W. Boyd (M'87) received the B.S. degree in physics in 1969 from the Massachusetts Institute of Technology, Cambridge, and the Ph.D. degree in physics in 1977 from the University of California, Berkeley. His dissertation work involved the use of nonlinear optical techniques for infrared detection for astronomy.

He joined the faculty of the Institute of Optics, University of Rochester, Rochester, NY, in 1977 and since 1987 has held the position of Professor of optics. His research interests include studies of the nonlinear optical properties of materials, nonlinear optical interactions in atomic vapors, optical phase conjugation, and the development of new laser systems.